



THEOREM OF THE DAY

The Existence Theorem for Orthogonal Diagonal Latin Squares *There exists, for every positive integer $n \neq 2, 3$ or 6 , a pair of orthogonal diagonal latin squares of order n .*

a	0	9	4	6	1	7	5	8	2	3
	7	1	9	4	5	3	8	0	6	2
	4	6	2	8	3	1	7	5	9	0
	6	0	7	3	2	8	4	9	1	5
	5	3	6	7	4	2	9	1	0	8
	8	4	1	2	9	5	0	6	3	7
	2	5	3	0	8	9	6	4	7	1
	3	2	8	9	0	4	1	7	5	6
	9	7	5	1	6	0	3	2	8	4
	1	8	0	5	7	6	2	3	4	9

Which pairs from this collection of diagonal latin squares of order 10 are orthogonal?

	0	4	1	9	8	2	7	3	5	6	b
	3	1	6	8	2	9	4	5	0	7	
	6	5	2	4	9	0	3	8	7	1	
	1	8	5	3	7	4	9	0	6	2	
	9	2	0	5	4	7	8	6	1	3	
	8	6	3	7	1	5	0	9	2	4	
	4	0	7	2	5	3	6	1	9	8	
	2	9	4	1	6	8	5	7	3	0	
	7	3	9	6	0	1	2	4	8	5	
	5	7	8	0	3	6	1	2	4	9	

c	0	6	8	1	9	7	3	4	2	5
	4	1	3	0	5	8	9	2	7	6
	6	9	2	5	8	4	7	1	0	3
	9	8	0	3	2	1	4	5	6	7
	3	7	1	8	4	6	0	9	5	2
	1	2	6	7	0	5	8	3	9	4
	7	4	5	2	1	9	6	8	3	0
	5	0	9	4	6	3	2	7	1	8
	2	3	4	9	7	0	5	6	8	1
	8	5	7	6	3	2	1	0	4	9

d	0	8	5	1	7	3	4	6	9	2
	5	1	7	2	9	8	0	3	4	6
	1	7	2	9	5	6	8	0	3	4
	9	6	4	3	0	2	7	1	5	8
	3	0	8	6	4	1	5	9	2	7
	4	3	0	8	6	5	9	2	7	1
	7	2	9	5	1	4	6	8	0	3
	6	4	3	0	8	9	2	7	1	5
	2	9	6	4	3	7	1	5	8	0
	8	5	1	7	2	0	3	4	6	9

Squares discovered by John Wesley Brown, Fred Cherry, Lee Most, Mel Most, Ernest T. Parker and Walter Wallis ("Completion of the spectrum of orthogonal diagonal Latin squares," in *Graphs, Matrices and Designs*, Dekker, 43–49, 1992)

	0	3	4	2	8	1	5	9	7	6	e
	2	1	6	8	3	0	7	5	9	4	
	9	5	2	0	6	7	4	3	1	8	
	4	7	9	3	0	6	1	8	5	2	
	7	0	5	9	4	8	3	6	2	1	
	8	9	1	6	7	5	2	4	3	0	
	3	8	7	4	9	2	6	1	0	5	
	1	6	0	5	2	9	8	7	4	3	
	6	2	3	1	5	4	9	0	8	7	
	5	4	8	7	1	3	0	2	6	9	

A diagonal latin square of order n is an $n \times n$ array with every integer from 0 to $n - 1$ in every row, every column and both main diagonals. Two such, $A = (a_{ij})_{1 \leq i, j \leq n}$, and $B = (b_{ij})_{1 \leq i, j \leq n}$, are *orthogonal* if taking all pairs of corresponding elements: $\{(a_{ij}, b_{ij}) \mid 1 \leq i, j \leq n\}$, gives precisely the set of all ordered pairs from $\{0, \dots, n - 1\}$. In 1779 Leonhard Euler observed that, in this case, $A + nB$ would make a magic square, since every row, column and diagonal would necessarily add to the same number: $n(n^2 - 1)/2$. In fact *any* two of the above latin squares combine to form a magic square in this way — but only three orthogonal pairings may be made.

Euler famously proposed that no order n pair of orthogonal latin squares existed for n of the form $4k + 2$, a conjecture demolished in 1960 by R. C. Bose, E. T. Parker and S. S. Shrikhande. Appropriately Parker was one of the team who, in 1992, discovered the order 10 orthogonal diagonal squares above, 10 being the last order for which existence of such pairs remained conjectural. This completed a proof whose construction had already spanned over twenty years and incorporated contributions by some of the leading experts in the field: Ervin Gergely, Katherine Heinrich, Anthony Hilton, Walter Wallis and Zhu Lie.

Web link: eulerarchive.maa.org/, Eneström Index E530 — where it all started.

Further reading: *Combinatorial Designs and Tournaments* by Ian Anderson, Clarendon Press, 1997.

