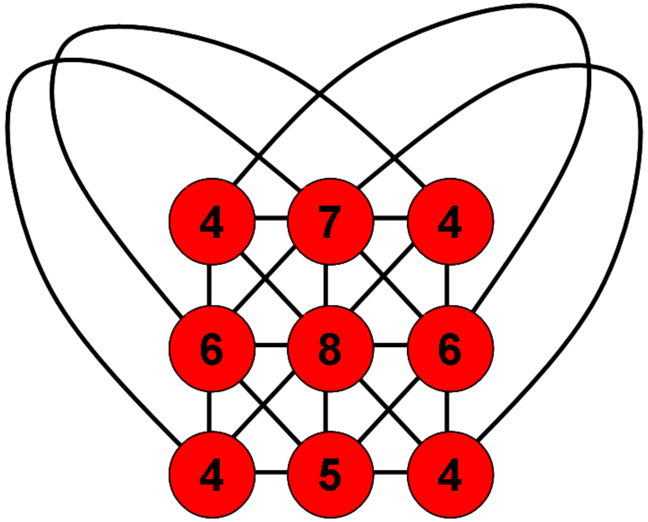
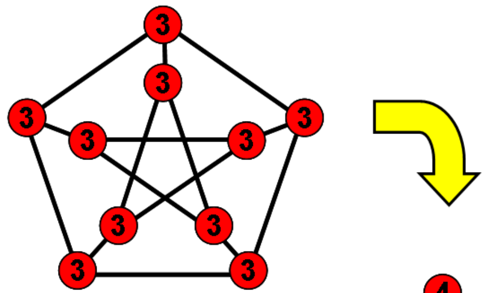
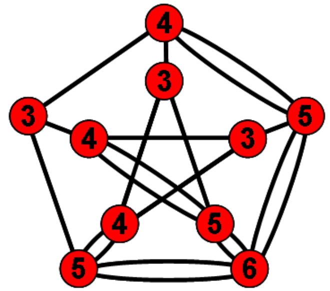
THEOREM OF THE DAY

The 1-2-3 Conjecture (a Theorem Under Construction!) *Call a graph, G,* degree colourable *if no two adjacent vertices have the same degree. If, by adjusting edge multiplicities in G so that no edge multiplicity exceeds k, we can create a degree colourable graph, then say that G is k-*degree colourable. *Now there exists a positive integer K, not dependent on the size of G, such that any connected graph on at least three vertices is K-degree colourable.*

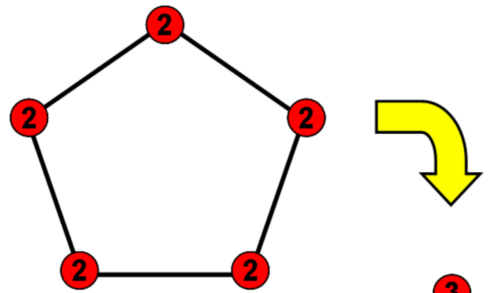


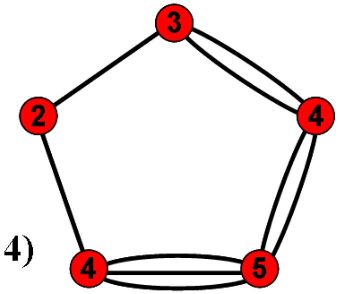
A graph based on the affine plane of order 3: $k = 1$





The Petersen graph: $k = 2$





The n-cycle: $k = 3$ unless $n \equiv 0 \pmod 4$

The degree of a vertex is the number of edges incident with that vertex. Above left, a graph based on (a representation of) the affine plane of order 3 is already properly coloured by its degrees. In the middle, the so-called Petersen graph is clearly not, since it is 3-regular (every vertex has degree 3) but doubling five suitably chosen edges shows that it is 2-degree colourable. Finally, doubling edges in the n-cycle is not sufficient, unless n is a multiple of 4, but degree colourability is 3: tripling edges is always enough.

Construction notes: 2002: Michal Karoński, Thomasz Łuczak and Andrew Thomason conjecture that K exists and equals 3. Prove it for 3-colourable graphs. The conjecture acquires the name "1-2-3 Conjecture".

2004: Louigi Addario-Berry, Ketan Dalal, Colin McDiarmid, Bruce Reed and Thomason: K exists and K=30 is sufficient.

2004: Addario-Berry, Dalal and Reed: reduce K to 16. (In 2008: they show $k \leq 2$ for 'almost all' graphs).

2008: Tao Wang and Qinglin Yu reduce K to 13.

2009: Maciej Kalkowski, Michał Karoński and Florian Pfender reduce K to 5.

2011: Andrzej Dudek and David Wajc: it is NP-complete to decide if $k \leq 2$ for a given graph.

Web link: arxiv.org/abs/1211.5122

Further reading: *Graph Theory (4th Edition)* by Reinhard Diestel, Springer New York, 2010.