**THEOREM OF THE DAY**

**The 1-2-3 Conjecture (a Theorem Under Construction!)**

Call a graph, $G$, degree colourable if no two adjacent vertices have the same degree. If, by adjusting edge multiplicities in $G$ so that no edge multiplicity exceeds $k$, we can create a degree colourable graph, then say that $G$ is $k$-degree colourable. Now there exists a positive integer $K$, not dependent on the size of $G$, such that any connected graph on at least three vertices is $K$-degree colourable.

A graph based on the affine plane of order 3: $k = 1$

The degree of a vertex is the number of edges incident with that vertex. Above left, a graph based on (a representation of) the affine plane of order 3 is already properly coloured by its degrees. In the middle, the so-called Petersen graph is clearly not, since it is 3-regular (every vertex has degree 3) but doubling five suitably chosen edges shows that it is 2-degree colourable. Finally, doubling edges in the $n$-cycle is not sufficient, unless $n$ is a multiple of 4, but degree colourability is 3: tripling edges is always enough.

**Construction notes:**

2002: Michal Karoński, Thomasz Łuczak and Andrew Thomason conjecture that $K$ exists and equals 3. Prove it for 3-colourable graphs. The conjecture acquires the name “1-2-3 Conjecture”.

2004: Louigi Addario-Berry, Ketan Dalal, Colin McDiarmid, Bruce Reed and Thomason: $K$ exists and $K=30$ is sufficient.

2004: Addario-Berry, Dalal and Reed: reduce $K$ to 16. (In 2008: they show $k \leq 2$ for ‘almost all’ graphs).

2008: Tao Wang and Qinglin Yu reduce $K$ to 13.

2009: Maciej Kalkowski, Michał Karoński and Florian Pfender reduce $K$ to 5.

2011: Andrzej Dudek and David Wajc: it is NP-complete to decide if $k \leq 2$ for a given graph.

**Web link:** arxiv.org/abs/1211.5122