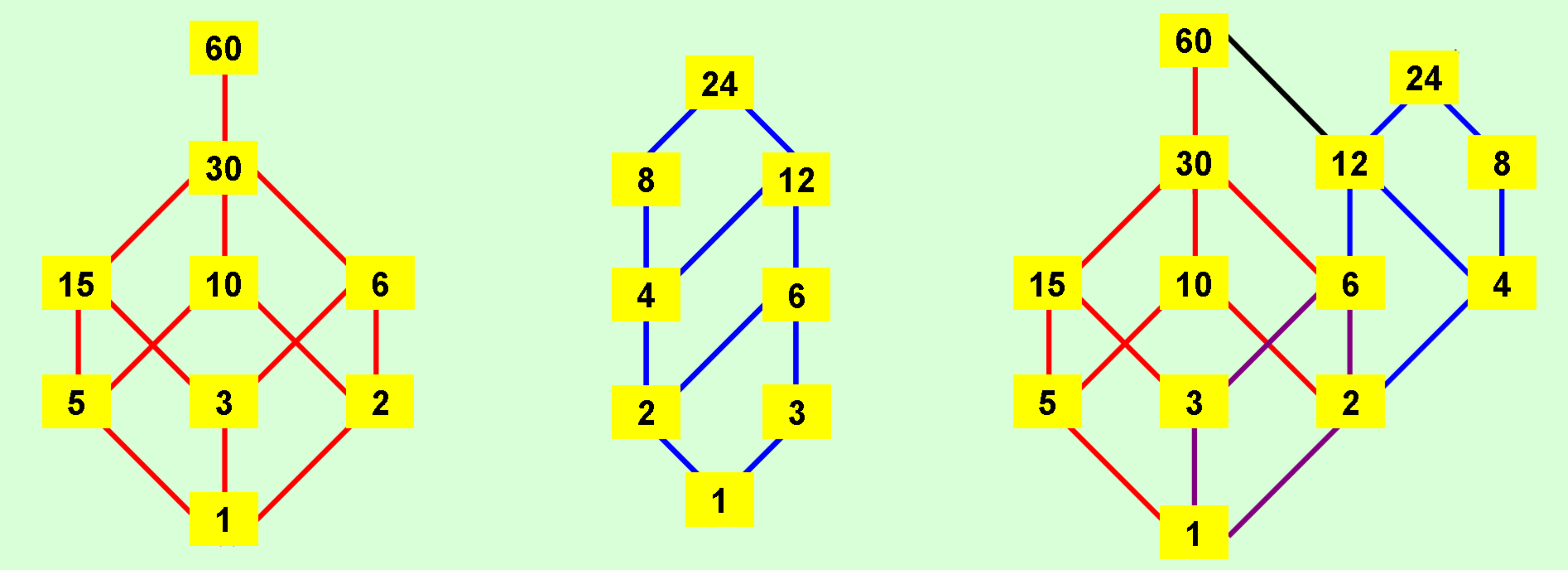
THEOREM OF THE DAY

Dilworth's Theorem *In a finite partial order, the size of a maximum antichain is equal to the minimum number of chains needed to cover all elements of the partial order.*



An ordering $\leq$ of the elements of a set is a *partial order* if, for any three elements, x, y and z, we have

Reflexivity: $x = y \Rightarrow x \leq y$; **Antisymmetry:** $x \neq y$ and $x \leq y \Rightarrow y \nleq x$; **Transitivity:** $x \leq y$ and $y \leq z \Rightarrow x \leq z$.

It is perfectly possible for a pair of elements, x and y, to satisfy both $x \nleq y$ and $y \nleq x$; such a pair is said to be *incomparable*. If the integers are partially ordered by the relation "divides exactly into", for example, then we have both $2 \nleq 3$ and $3 \nleq 2$. This is the ordering in the three partially ordered sets (*posets*) shown above, but note that not all pairs that *are* comparable are joined by an edge: from $1 \leq 2$ and $2 \leq 6$, for instance, transitivity automatically implies $1 \leq 6$. These minimally completed diagrams, in which the ordering progresses upwards, are called *Hasse diagrams*.

An *antichain* is any set of mutually incomparable elements; a *chain* is any set of mutually comparable ones (which includes all ascending paths in a Hasse diagram). Thus, $\{3, 8, 10\}$ is an antichain in the Hasse diagram, above right; $\{8, 10, 12, 15\}$ is a maximum antichain of size 4; and $\{\{5, 15\}, \{1, 10, 30, 60\}, \{3, 6, 12\}, \{2, 4, 8, 24\}\}$ is a set of 4 chains which cover all elements.

Robert Dilworth's 1950 theorem may be restated thus: if a poset has $ab + 1$ elements then it has a chain of length $a + 1$ or an antichain of length $b + 1$. In this form it generalises a classic 1935 result of Erdős and Szekeres. An interesting Peter Cameron footnote explains that Dilworth's theorem was "found a few years earlier by Gallai and Milgram, but publication was delayed because Gallai wanted the paper translated into English, and Milgram, a topologist, did not fully appreciate its importance."

Web link: www.math.cmu.edu/~af1p/Teaching/Combinatorics/Slides/Posets.pdf

Further reading: *Combinatorics: Topics, Techniques, Algorithms* by Peter Cameron, Cambridge University Press, 1994, chapter 12.