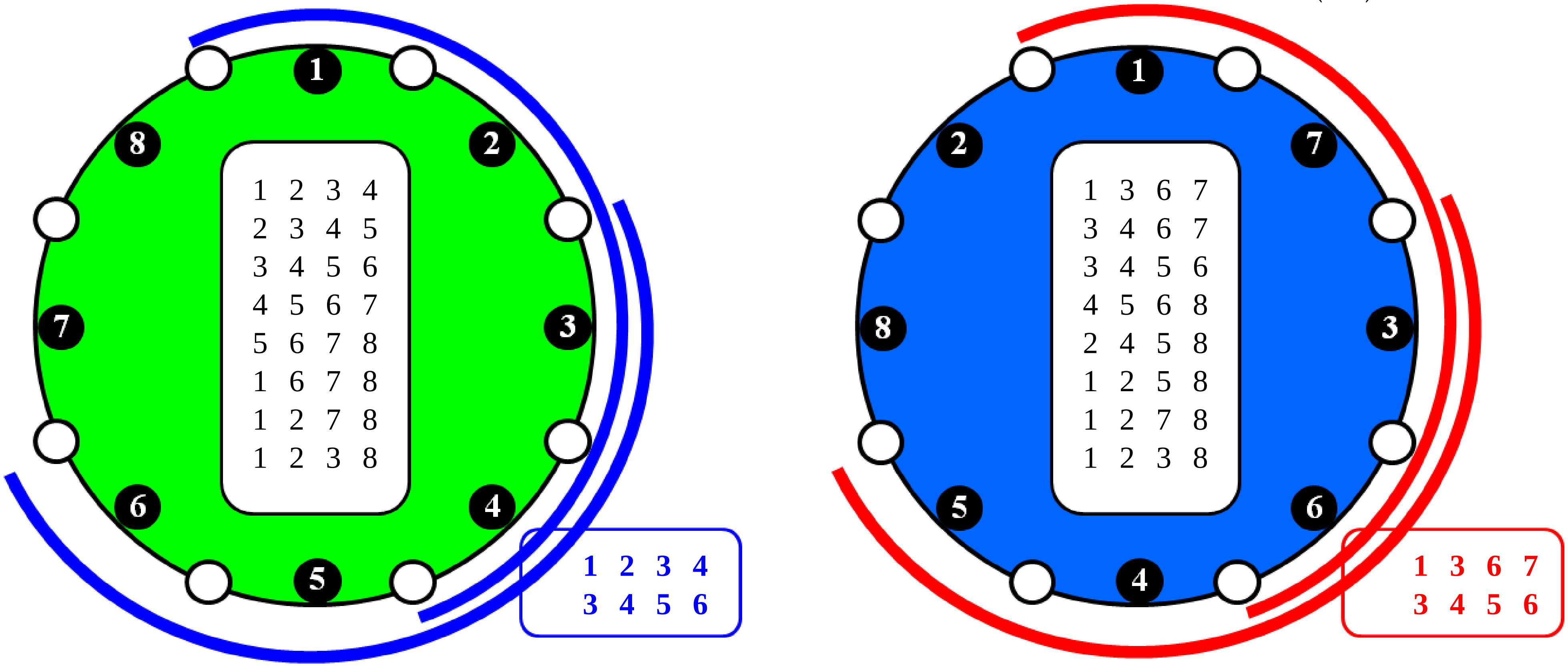
THEOREM OF THE DAY

The Erdős–Ko–Rado Theorem *Let n and k be positive integers, with $n \geq 2k$. In a set of cardinality n, a family of distinct subsets of cardinality k, no two of which are disjoint, can have at most $\binom{n-1}{k-1}$ members.*



In our illustration, $n = 8$ and $k = 4$. Following the ingenious 1972 proof by Gyula O.H. Katona, we arrange permutations of $1, \ldots, 8$ cyclically: fixing 1 there are $(n-1)! = 7!$ such arrangements; two are shown above. If they label the edges of the 8-cycle, as shown, then a sequence of four consecutive edges produces a subset of $\{1, 2, \ldots, 8\}$ of cardinality 4. And we can find at most $k = 4$ such subsets satisfying the condition that any two overlap in at least one edge: for the cycle, above left, there are a total of eight sets corresponding to consecutive four-edge sequences; two are highlighted, and two more can be added before we start getting disjoint sets. On the right, the whole thing is repeated but using a different permutation.

Summing over all $(n-1)!$ cyclic permutations, each contributing k overlapping sets, we see that we cannot get more than a total of $k(n-1)!$ overlapping sets. But certainly this will involve double counting some sets: between the two permutations shown above we have already counted $\{3, 4, 5, 6\}$ twice. Luckily we can count how often this happens: each set has $k!$ arrangements leaving $(n-k)!$ arrangements to complete a permutation. So a family of sets can be 'intersecting' — no pair disjoint — with at most $k(n-1)!/k!(n-k)! = \binom{n-1}{k-1}$ members. Note that the bound is trivially achieved by joining $\{n\}$ to every $(k-1)$-subset of $\{1, \ldots, n-1\}$.

Paul Erdős, Chao Ko and Richard Rado proved this fundamental theorem of extremal set theory in 1938.

Web link: www.fq.math.ca/48-2.html: the paper by Butler, Horn and Tressler.

Further reading: *Extremal Combinatorics with Applications in Computer Science* by Stasys Jukna, Springer, 2nd edition, 2011, chapter 7.