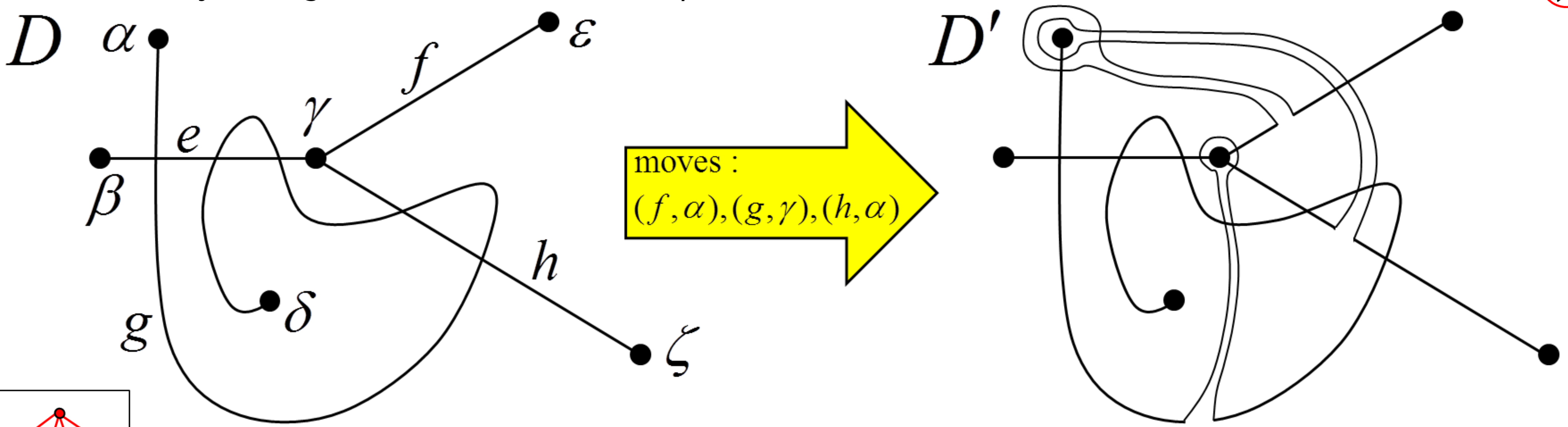
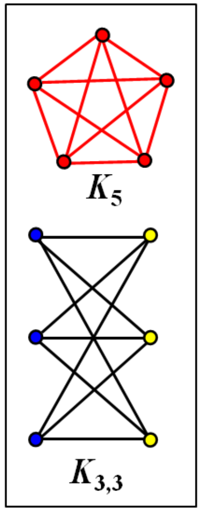
THEOREM OF THE DAY

The Hanani–Tutte Theorem *A graph is planar if and only if it has a drawing in which all pairs of vertex-disjoint edges cross an even number of times.*





A planar graph is one which has a drawing in the plane with zero edge crossings, so the ‘only if’ part of this theorem is clear. It can be proved that any drawing of K_5 or K_{33} must contain a pair of vertex-disjoint edges which cross an odd number of times: we will call this an ‘odd independent edge pair’. Moreover, this property is preserved if some edges are subdivided to become paths, and if additional edges are added. So by Kuratowski’s theorem, any nonplanar graph contains an odd independent edge pair: the (contrapositive of the) ‘if’ part of the theorem.

A remarkable corollary of this theorem is an algebraic test for planarity. Some notation: suppose D is a drawing of G and that we denote by $i_D(e, f)$ the number of times, reduced modulo 2, that edges e and f cross in D. The ‘if’ part of the theorem then says that, for some drawing D, for all vertex-disjoint edges e and f we have $i_D(e, f) = 0$. Now in drawing D, for a vertex v and edge e, define an ‘(e, v)-move’ to be a smooth deformation of e which passes across vertex v but no other vertices. The prototypical form of such a deformation is shown in the above figure: a thin ‘offshoot’ of the edge which reaches just beyond the vertex. Its crucial effect is to add one, modulo 2, to $i_D(e, f)$ if and only if edge f is incident with v. You can check this in the above examples: move (h, α), for example, increases the number of crossings of h with g (incident with α) from 2 to 5 (change of parity). The number of crossings of h with f (non-incident with α) changes from 0 to 2 (no change of parity). Our algebra will be done with variables $x_{e,v}$ which take binary (mod 2) values. We will use $x_{e,v} = 1$ to mean we apply an (e, v)-move and $x_{e,v} = 0$ to mean we do not apply the move.

Now give each edge of G an arbitrary orientation, so that any edge e has a ‘tail’ e^- and a head e^+. Then for edges e and f, observe that the equality

$$i_D(e, f) + x_{e,f^-} + x_{e,f^+} + x_{f,e^-} + x_{f,e^+} = i_{D'}(e, f), \qquad (1)$$

holds, for any choice of values of the variables, where D' is the drawing which results from the chosen moves. In our example, above, orienting edges generally left-to-right and top-to-bottom, we have for instance, $x_{e,g^-} + x_{e,g^+} + x_{g,e^-} + x_{g,e^+} = x_{e,\alpha} + x_{e,\delta} + x_{g,\beta} + x_{g,\gamma} = 0 + 0 + 0 + 1$, so $i_{D'}(e, g) = i_D(e, g) + 1 = 1 + 1 = 0$.

And we have our algebraic planarity test: start with an oriented drawing D of G; write equation (1) for every pair of independent edges; then G is planar if and only the system of equations, all set equal to zero, have a solution modulo 2.

The theorem is named after Haim Hanani (1934) and Bill Tutte (1970) who rediscovered it and invented the algebraic setting.

Web link: www.emis.de/journals/JGAA/issues-20.html: the paper (vol. 17, no. 4) by Marcus Schaefer.
Further reading: *A Course in Topological Combinatorics* by Mark de Longueville, Springer, 2012.