THEOREM OF THE DAY

Woodall’s Hopping Lemma Among the maximum-length cycles in a graph G, choose one which, when its vertices are deleted, breaks G into the smallest possible number of connected components, and suppose that one such component is a single vertex v. Let the chosen cycle be \( C = (v_0, \ldots, v_{t-1}, v_0) \) (the \( v_i \) being indexed modulo \( t \)). Now define two collections of vertex sets by:

\[
Y_0 = \emptyset; \quad \text{and for } i = 1, 2, \ldots, X_i = N(Y_{i-1} \cup \{v\}), \quad \text{and } Y_i = \{v_j \in C \mid v_{j-1}, v_{j+1} \in X_i\},
\]

where for a vertex set \( S \), \( N(S) \) is the set of all vertices adjacent to vertices in \( S \). Then for all \( i, j \geq 1 \), \( X_i \) is a subset of \( C \) which is disjoint from \( Y_j \) and contains no two consecutive vertices of cycle \( C \).

Let graph \( G \) have vertex set \( V \) and consider \( b(G) = \min |N(S)|/|S| \), minimising over nonempty, proper subsets \( S \) of \( V \) for which \( N(S) \neq V \). In 1973, Douglas Woodall proved that if \( b(G) \geq 3/2 \) then \( G \) is Hamiltonian, inventing his Lemma for the purpose.

It has since found and continues, in many variants, to find, numerous related applications.