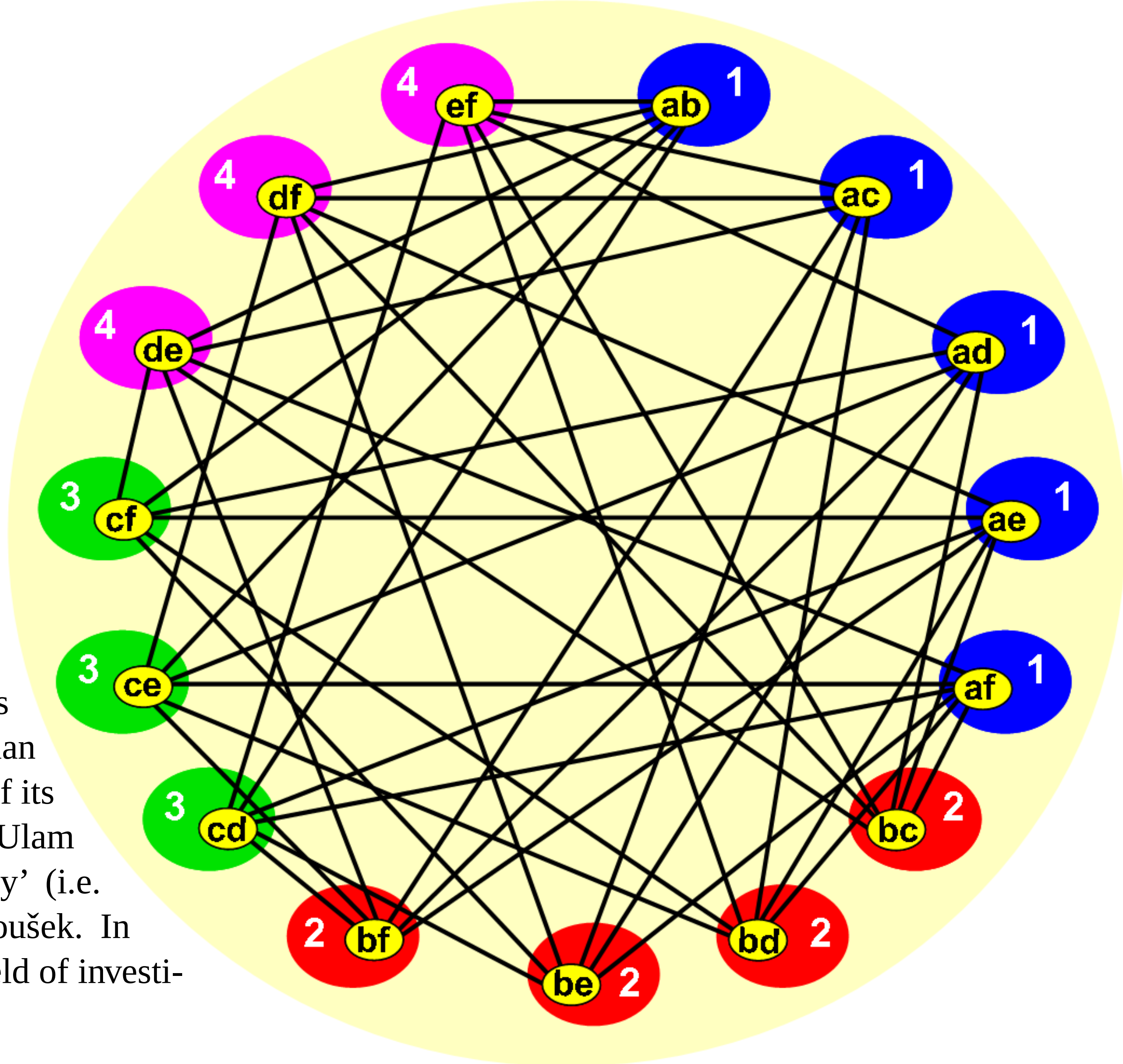
THEOREM OF THE DAY

Kneser's Conjecture *For positive integers n and k, $k \leq (n+1)/2$, let $C_{n,k}$ denote the set of all k-element subsets of $\{1,\ldots,n\}$. Now for positive integer t, let $C_1 \cup \ldots \cup C_t = C_{n,k}$ be a partition of $C_{n,k}$ such that any two sets in any of the C_i intersect nontrivially (i.e., $c, c' \in C_i \Rightarrow c \cap c' \neq \emptyset$, for $1 \leq i \leq t$). Then $t \geq n - 2k + 2$.*

Denote by $[n]$ the set $\{1,\ldots,n\}$. The property of $[n]$ asserted by this theorem is viewed in a natural way in terms of graph colouring. Define the *Kneser graph* $KG_{n,k}$ (pron. K-**nay**-zer) by taking the $\binom{n}{k}$ subsets of $[n]$ of size k as vertices, joining two by an edge precisely when they are disjoint. The graph $KG_{6,2}$ is shown on the right, with [6] represented by the letters 'a',...,'f'. Now, a t colouring of $KG_{n,k}$ partitions the vertices into t colour classes so that no edge joins vertices of the same colour class. The theorem says that the smallest value of t, that is, the *chromatic number* of $KG_{n,k}$, is $n - 2k + 2$. When $n = 6$ and $k = 2$, this gives a value of 4, and the 4-colouring of $KG_{6,2}$ shown as large numbered ovals on the right can readily be seen to extend to an $(n - 2k + 2)$-colouring of $KG_{n,k}$ in the general case. An $(n - 2k + 1)$-colouring, however, is never possible.



Martin Kneser (1928–2004) proposed this property of set systems in 1955, in connection with a study of quadratic forms. Apart from its inherent interest, its eventual proof, in 1978 by the Hungarian mathematician László Lovász, sparked enormous interest because of its reliance on a deep theorem of topology, the Borsuk-Ulam theorem. Not until 2000 did a difficult but 'elementary' (i.e. purely combinatorial) proof appear, due to Jiřì Matoušek. In the meantime, Lovász's work had opened up a new field of investigation: topological combinatorics.

Web link: www.emis.de/newsletter/current/ (pages 16–19)

Further reading: *Using the Borsuk-Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry* by Jiřì Matoušek, Springer, Berlin, 2003.