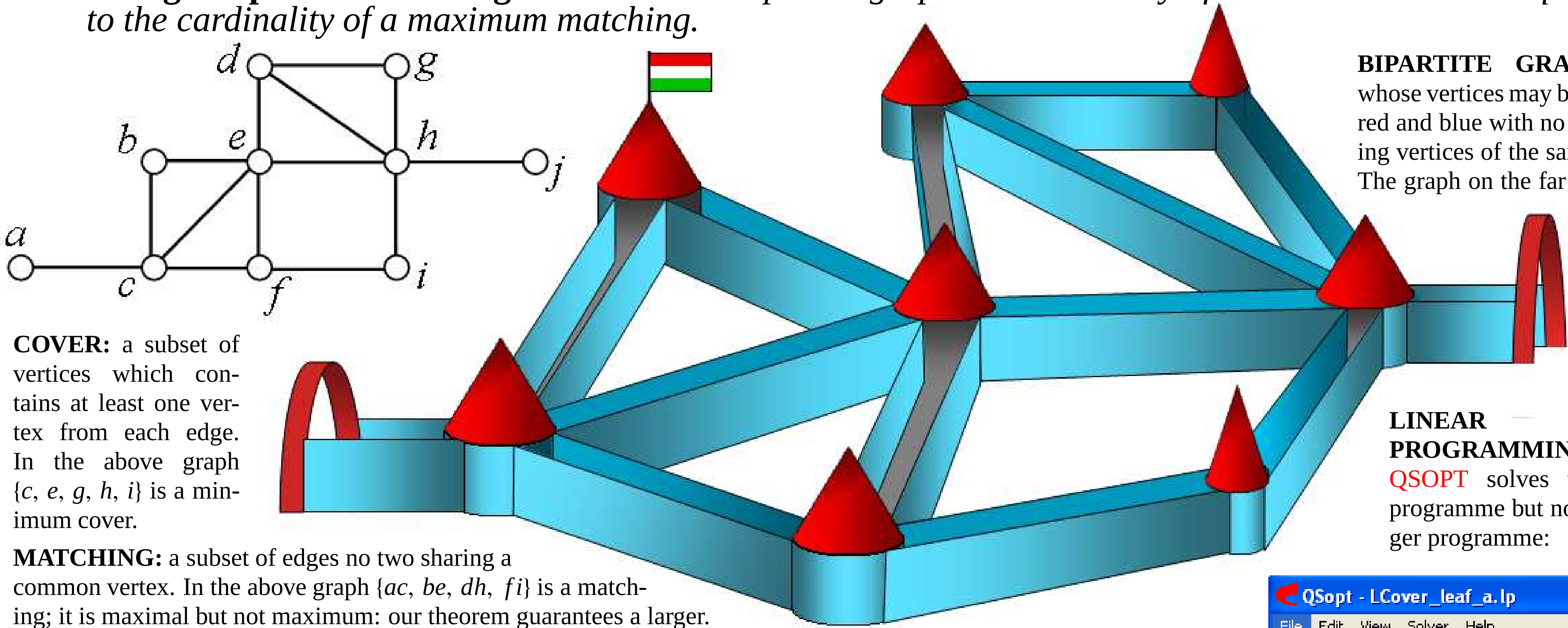
THEOREM OF THE DAY

Kőnig's Bipartite Matching Theorem *In a bipartite graph the cardinality of a minimum cover is equal to the cardinality of a maximum matching.*

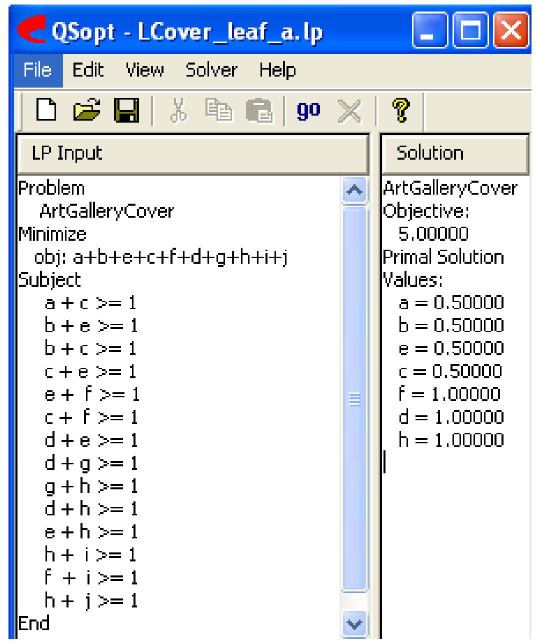


BIPARTITE GRAPH: one whose vertices may be coloured red and blue with no edge joining vertices of the same colour. The graph on the far left is **not** bipartite.

COVER: a subset of vertices which contains at least one vertex from each edge. In the above graph $\{c, e, g, h, i\}$ is a minimum cover.

MATCHING: a subset of edges no two sharing a common vertex. In the above graph $\{ac, be, dh, fi\}$ is a matching; it is maximal but not maximum: our theorem guarantees a larger.

LINEAR PROGRAMMING: QSOPT solves the linear programme but not the integer programme:



MINIMAX AND DUALITY: what is the least number of surveillance cameras needed to monitor the corridors of the art gallery depicted above? The graph model, top left, formulates this as a minimum cover problem: the vertices represent the towers where corridors meet; the edges are the corridors. We can go further: we let non-negative integers represent the vertices; constraints such as $a + c \geq 1$ encode the requirement that at least one vertex of each edge must be non-zero (i.e., in the cover); minimising the sum of the vertices selects a minimum set satisfying all the constraints (see the QSOPT example on the right). Alas, for such **integer programmes** no effective solution method is known; if we ask only that vertex values are non-negative *reals* then we have the 'relaxation' to a **linear programme**, which *is* effectively solvable. But non-integer solutions (as occurs in our QSOPT example) will be meaningless. Can we rescue something? Yes! Here is the double punch line: (1) for *bipartite* graphs, our linear programme is guaranteed to give only integer solutions; and (2) the **dual programme**, maximising an edge sum subject to dual constraints such as $ac + bc + ce + cf \leq 1$, will find a maximum matching whose cardinality must, by linear programming duality, equal that of our minimum cover solution. Thus ...

... Dénes Kőnig's 1931 'minimax' theorem, the culmination of his pioneering work in graph theory.

Web link: www.theoremoftheday.org/Docs/RWLectureNotes/GraphAlgorithms/Ch9.html
Some history: homepages.cwi.nl/~lex/files/histco.pdf.

Further reading: *Combinatorial Optimization* by C. Papadimitriou and K. Steiglitz, Dover Publications, 2000.