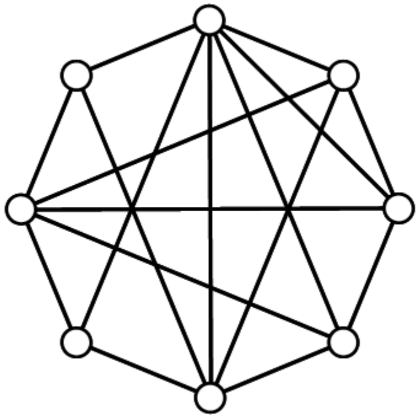
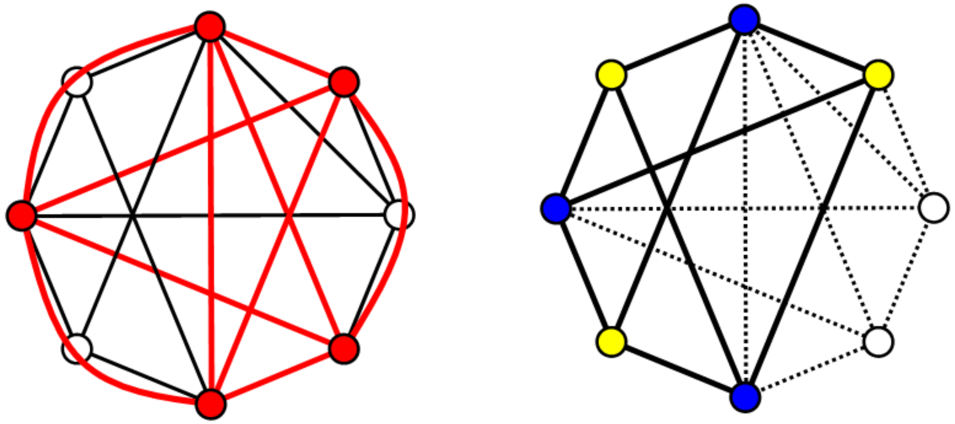
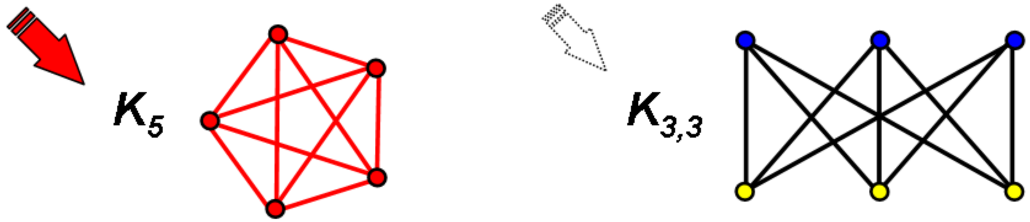
THEOREM OF THE DAY

Kuratowski's Theorem *A graph G is planar if and only if it contains neither K_5 nor $K_{3,3}$ as a topological minor.*







Is the graph on the left *planar*? That is, can it be redrawn so that edges only intersect each other at one of the eight vertices? Confirming a positive answer is potentially easy: just show the redrawing. But a proof that there is *no* redrawing seems elusive: we might try a million unsuccessful attempts but feel we *nearly* have it. By an easy application of Euler's Polyhedral Formula, a graph with n vertices and more than $3n-6$ edges cannot be planar. But this does not apply to the graph above left which has $17 < 3\times 8-6$ edges. Kuratowski's theorem comes to the rescue: nonplanarity is confirmed as soon as we exhibit either K_5 or $K_{3,3}$ as a *topological minor*. H is a topological minor of G if it appears as a subgraph of G but with its edges replaced by internally disjoint paths (edge-sequences which share only their end points). In the picture, three of the edges of K_5 appear as paths of length two; $K_{3,3}$ is actually a subgraph (all the disjoint paths are just the original edges). In this graph *both* of the forbidden topological minors are present; of course, either one alone is enough to prevent planarity.

By the way, K_5 itself is nonplanar since its edge count is $10 > 3\times 5-6$. The nonplanarity of $K_{3,3}$ also follows from Euler's formula, e.g. see rip94550.wordpress.com/2008/11/30/.

The Polish mathematician Kazimierz Kuratowski proved his theorem in 1930. Forty years later the dedication of Frank Harary's classic *Graph Theory* was:

> To KASIMIR KURATOWSKI,
> Who gave K_5 and $K_{3,3}$
> To those who thought planarity
> Was nothing but topology.

(In fact three other almost simultaneous discoveries of the theorem are recorded: Orrin Frink and Paul Althaus Smith; Lev Semenovich Pontrjagin; and Karl Menger!)

Web link: www.cimt.org.uk/projects/mepres/alevel/alevel.htm, Discrete Mathematics Chapter 6; a short proof can be found here: ttic.uchicago.edu/~yury/papers/kuratowski.pdf; and for fun: planarity.net!

Further reading: *Graphs and Digaphs, 5th Edition* by Gary Chartrand, Linda Lesniak and Ping Zhang, Chapman and Hall/CRC, 2010, chapter 6.