THEOREM OF THE DAY

The LYM Inequality Let S be a finite set and \mathscr{F} a family of subsets of S none of which is a subset of

any other. Then

$$\sum_{X \in \mathscr{F}} \binom{|S|}{|X|}^{-1} \le 1.$$



Proof. The inequality may be proved from a more general 1965 result of Béla Bollobás:

Let *S* be a finite set of cardinality *n* and let A_i and B_i , $1 \le i \le m$, be sequences of subsets of *S* having the property that $A_i \cap B_j = \emptyset$ if and only if i = j, $1 \le i, j \le m$. Then $\sum_{i=1}^{m} {a_i + b_i \choose a_i}^{-1} \le 1$, where $a_i = |A_i|$ and $b_i = |B_i|$, i = 1, ..., m.

Indeed, in the LYM inequality, set $m = |\mathscr{F}|$ and list the sets of \mathscr{F} as A_1, A_2, \ldots, A_m . Now set $B_i = S \setminus A_i$ for all *i*. Then $A_i \cap B_j = A_i \cap (S \setminus A_j) = \emptyset$ if and only if $A_i \subseteq A_j$ if and only if i = j. To check a small example, consider the rows in the matrix depicted above which are labelled with four pairs of subsets A_i, B_j from the set $\{1, 2, 3, 4, 5\}$.

Proof of Bollobás's Inequality. By double counting: form a matrix whose rows are indexed by the pairs A_i , B_i , i = 1, ..., m, and whose columns are indexed by the *n*! permutations of $\{1, ..., n\}$ (the matrix above illustrates this with n = 5 and m = 4). Now set the *ij*-th entry of the matrix to 1 if, in the *j*-th permutation, all elements of set A_i are listed before all elements of set B_i . Set all other entries to zero. In the matrix above the entries set to 1 are shaded red. For example, the third entry in the first row is red because $A_1 = \{1, 2, 4\}, B_1 = \{3\}$ and the third permutation, (1, 2, 4, 3, 5) permutes 3 to come after 1, 2 and 4. There are exactly $\binom{n}{a_i+b_i}a_i!b_i!(n-(a_i+b_i))! = n!\binom{a_i+b_i}{a_i}^{-1}$ permutations which are shaded red in row *i*. And a permutation may be shaded red in at most one row since if

permutation π permutes A_i before B_i and A_j before B_j then $A_i \cap B_j = \emptyset$ or $A_j \cap B_i = \emptyset$, whence i = j. So $\sum n! \binom{a_i + b_i}{a_i}^{-1} \le n!$ and the inquality follows.

With |S| = n in the LYM inequality and using the fact that $\binom{|S|}{|X|} = \binom{n}{|X|} \le \binom{n}{|n/2|}$ we derive one of the foundational results of combinatorial set theory: **Corollary (Sperner's Theorem, 1928)** Let *S* be a finite set of cardinality *n*, and \mathscr{F} a family of subsets of *S* none of which is a subset of any other. Then $|\mathscr{F}| \le \binom{n}{\lfloor n/2 \rfloor}$. LYM is named for its independent discoverers: David Lubell (1966), Koichi Yamamoto (1954) and Lev Dmitrievich Meshalkin (1963).

Web link: lastinggems.wordpress.com/tag/bollobas/

Further reading: Combinatorics of Finite Sets by Ian Anderson, Dover reprint, 2003.