THEOREM OF THE DAY

Beineke’s Theorem on Line Graphs Let $G$ be a graph. There exists a graph $H$ such that $G$ is the line graph of $H$ if and only if $G$ contains no induced subgraph from the following set:

For a graph $H$, the line graph $L(H)$ has a vertex for every edge of $H$ and an edge for every pair of incident edges of $H$.

An induced subgraph is a subset of the vertices of $G$ together with each and every edge of $G$ joining any two vertices of this subset. The ‘very nonlinear’ graph above contains every one of Beineke’s forbidden set (see how quickly you can find them; the tricky bit is that they must be induced so that $a$, $b$, $d$, $h$, for example, will not do for graph $I$ whose three end-vertices cannot be adjacent to each other).

Line graphs are a fundamental construction in graph theory. For example, edge-colourings of a graph $G$, where incident edges must be differently coloured, are the same as vertex colourings of $L(G)$, where adjacent vertices are differently coloured. In this 1968 characterisation of line graphs, due to Lowell Beineke and, independently, Neil Robertson, the forbidden induced subgraph I is known as a ‘claw’. The property of being ‘claw-free’, possessed by line graphs, is again fundamental in graph theory.

Web link: [www.newton.ac.uk/event/csmw06/seminars](http://www.newton.ac.uk/event/csmw06/seminars): see Graphs and Matroids I, by Bill Jackson.


A dedication for today’s theorem:

To the mathematician Lowell Beineke,
Who characterised the graphs leineke
Thus: it is forbidden
In $G$ to find hidden
A graph from his set of size neineke.