# THEOREM OF THE DAY

The Matrix Tree Theorem *Let G be a graph with n vertices and let L(G) be the n × n matrix whose entry in row i and column j is defined to be*

$$-(\text{the number of edges joining vertex } i \text{ to vertex } j) \quad \text{if } i \neq j, \quad \text{and}$$
$$\text{the number of edges incident with vertex } i \quad \text{if } i = j.$$

Then the number of spanning trees of G is given by $\det L(G)(1|1)$, *where* $L(G)(1|1)$ *is the matrix obtained by deleting the 1st row and 1st column of L(G).*
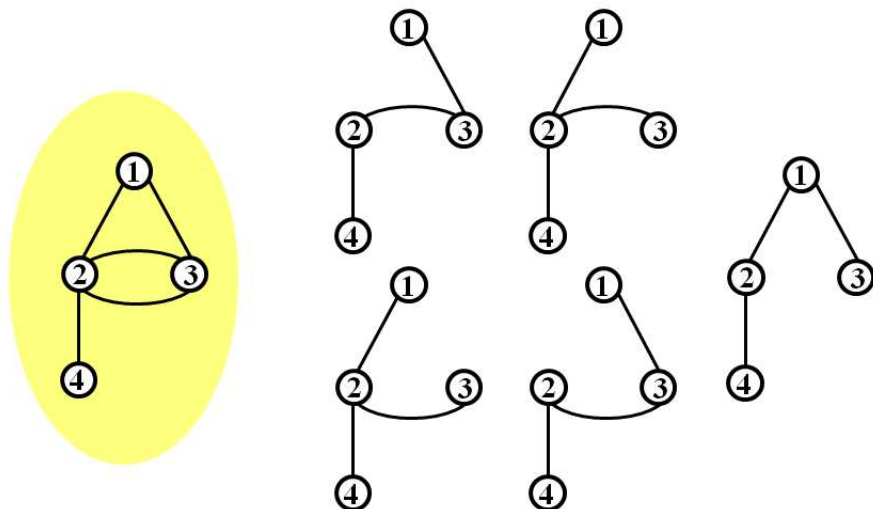
A 4-vertex, 5-edge graph is shown above left. Its spanning trees, shown in the centre, are those connected subsets of edges which are incident with all 5 vertices and which contain no cyclic paths. They necessarily have $4 - 1 = 3$ edges. Notice that these spanning trees are distinct as **labelled** objects: there are only two 'structurally' different trees.

An Application
How 'reliable' is the graph on the left? We could interpret this to mean, say, what is the probability that deleting two edges at random from the graph on the left will still leave a connected graph? Removing two edges gives $\binom{5}{2} = 10$ subgraphs on three edges. Of these, the only subgraphs that are connected are the five spanning trees. So our probability is $\frac{5}{10} = \frac{1}{2}$.

The *determinant* function, det, yields a single figure from a square matrix or table. It is available in standard spreadsheet applications as the '=MDETERM' function. As shown here in the OpenOffice Calc package, the calculation will produce the answer 5, since there are exactly 5 spanning trees for the given graph. In fact, any row of L(G), not just the first, and any column, may be deleted in the statement of the theorem without changing the absolute value of the result.

The determinant function predates the development of matrix theory (initiated by Cayley in the 1850s) by nearly two hundred years and this tree-counting calculation was first devised by Gustav Kirchhoff in 1847 as a way of obtaining values of current flow in electrical networks.

Web link: www.math.ku.edu/~jmartin/mc2004/graph1.pdf. See Garrys Tee's article in vol. 30 of Image for more on the history and applications of determinants.

Further reading: *Introduction to Graph Theory, 5th Ed.,* by Robin J. Wilson, Prentice Hall, 2010, Chapter 4.