THEOREM OF THE DAY

The Ollerenshaw–Brée Formula *Let n, a positive integer, have prime factorisation $n = p_1^{s_1} p_2^{s_2} \cdots p_r^{s_r}$ in which $p_1 = 2$ and $s_1 \geq 2$, so that n is doubly even. The number of $n \times n$ most-perfect magic squares, up to horizontal, vertical and diagonal symmetry, is given by*

$$N(n) = M(n) \sum_{v=0}^{\tau(n)} W(v)[W(v) + W(v+1)],$$

where $\tau(n)$ is the number of divisors of n and $M(n)$ and $W(v)$ are given by $M(n) = 2^{n-2}(n/2)!^2$, and $W(v) = \sum_{i=0}^{v} (-1)^{v+i} \binom{v+1}{i+1} \prod_{j=1}^{r} \binom{s_j+i}{i}$.

Suppose the integers $0, \ldots, n^2-1$ are arranged in an $n \times n$ array, n a multiple of 4. The result is called *most-perfect* if (a) any two entries at distance $n/2$ on any diagonal sum to n^2-1; and (b) the entries in any 2×2 block of adjacent cells sums to $2(n^2-1)$. In the 8×8 example here, the cells diagonally opposite on the yellow bands (sloping down to the right) are pairwise at distance 4 and sum, pairwise, to 63. The 2×2 blocks, including 'broken' blocks, like the one which counts the top-left cell as adjacent to bottom-left and top-right and so on, have cells summing to 126. Most-perfect squares are *pandiagonally magic*: each complete row, column and (broken) diagonal (such as the yellow and green diagonals) sum to the same quantity: $n(n^2-1)/2$. The 4×4 example shows how a most-perfect square may be constructed from any *reversible square*, that is, one in which all 2×2 *submatrices* have equal diagonal sums, and in which, in any row or column, the sums (cell i) + (cell $n-i+1$) are equal for all i, $1 \leq i \leq n/2$. Conversely, *every* most-perfect square may be obtained in this way.

A most-perfect 8×8 square

0	62	2	60	7	57	5	59
15	49	13	51	8	54	10	52
16	46	18	44	23	41	21	43
31	33	29	35	24	38	26	36
56	6	58	4	63	1	61	3
55	9	53	11	48	14	50	12
40	22	42	20	47	17	45	19
39	25	37	27	32	30	34	28

Kathleen Ollerenshaw was in her 80s when she made the amazing discovery that a 1938 construction of J. Barkley Rosser (of the Church-Rosser property) and Robert J. Walker could be adapted to give precisely the members of this major class of magic squares. Working with David Brée she exploited this to give the first ever counting formula for *any* such class (the total number of all $n \times n$ magic squares is unknown even for $n = 6$).

An example of order 4

0	1	4	5
2	3	6	7
8	9	12	13
10	11	14	15

Reverse column order on RHS

0	1	5	4
2	3	7	6
8	9	13	12
10	11	15	14

Reverse row order in bottom half

0	1	4	5
2	3	6	7
10	11	15	14
8	9	13	12

Distribute into 2×2 blocks

0	14	5	11
7	9	2	12
10	4	15	1
13	3	8	6

Web link: recmath.org/Magic_Squares/most-perfect.htm

Further reading: *Most Perfect Pandiagonal Magic Squares*, by Kathleen Ollerenshaw and David Brée, Institute of Mathematics and its Applications, 1988.