The Pólya–Redfield Enumeration Theorem

Denote by $Z(G, \Omega; s_1, \ldots, s_t)$ the cycle index of a permutation group $G$ acting on a set $\Omega$. For a set of distinct invariants $L = \{x_1, \ldots, x_n\}$, denote by $x^{(k)}$, $k = 1, \ldots, t$, the formal sum $x_1^k + \ldots + x_n^k$. Then the number of different ways in which $\Omega$ may be labelled with elements of $L$ up to symmetry under the action of $G$ is enumerated by $Z(G, \Omega; x, x^{(2)}, \ldots, x^{(t)})$.

We construct the cycle index for $\Omega_V$, the set of vertices of the regular tetrahedron, and for $\Omega_E$, its set of edges. The group acting on these sets will be the rotational symmetries of the tetrahedron, which is the alternating group $A_4$. As an abstract group, $A_4$ may be defined as multiplication of the twelve even permutations of $\{1, \ldots, 4\}$:


(‘even’ by virtue of their numbers of disjoint cycles having the same parity as the permuted set, given that 1-cycles are listed, as above). If the vertices are labelled 1, 2, 3, 4, as above left, then our even permutations directly represent the rotational symmetries of the tetrahedron: for example, $(1 2)(3 4)$ is a half-rotation about an axis through the midpoints of edges $a$ and $d$. The cycle index encodes these symmetries in terms of their cycle lengths: $s_i$ records cycles of length $i$, hence $s_i^i$ records a permutation of four 1-cycles (i.e. $(1)(2)(3)(4)$); $3s_3^2$ records three pairs of 2-cycles; $8s_1s_3$ records eight permutations with a 1-cycle and a 3-cycle. And finally $Z(A_4, \Omega_V; s_1, s_2, s_3)$ averages these over the whole group to give $\frac{1}{12}(s_1^4 + 3s_2^2 + 8s_1s_3)$. Now we may enumerate, say, in how many ways the vertices may be coloured red, $r$, and blue, $b$, up to symmetry: $Z(A_4, \Omega_V; r + b, r^2 + b^2, r^3 + b^3) = b^4 + b^3r + b^2r^2 + br^3 + r^4$. The coefficient of $b^i r^j$ counts colourings with $i$ blue vertices and $j$ red ones: we see that the colouring of the tetrahedral above centre is unique, up to symmetry. When the tetrahedral symmetries act on edges instead of vertices the group is still $A_4$ but the disjoint cycle structures have changed and this is reflected in a different cycle index (see above left again; this time 1-cycles are omitted for conciseness):

$1, (a d)(b e), (a d)(c f), (b e)(c f), (a b c)(d e f), (a b f)(c d e), (a c b)(d e f), (a c e)(b d f), (a f e)(b d c), (a f e)(b d c) \to \frac{1}{12}(s_1^4 + 3s_2^2 s_3^2 + 8s_3^3)$.

This time the calculation reveals that the half-red-half-blue colouring, above right, is one of four possibilities up to symmetry.

The 1937 rediscovery by George Pólya of this neglected 1927 result of J. Howard Redfield turned it into a classic of combinatorics.

Web link: crypto.stanford.edu/pbc/notes/; the Redfield story: match.pmf.kg.ac.rs/content46.htm, the article by E. Keith Lloyd.