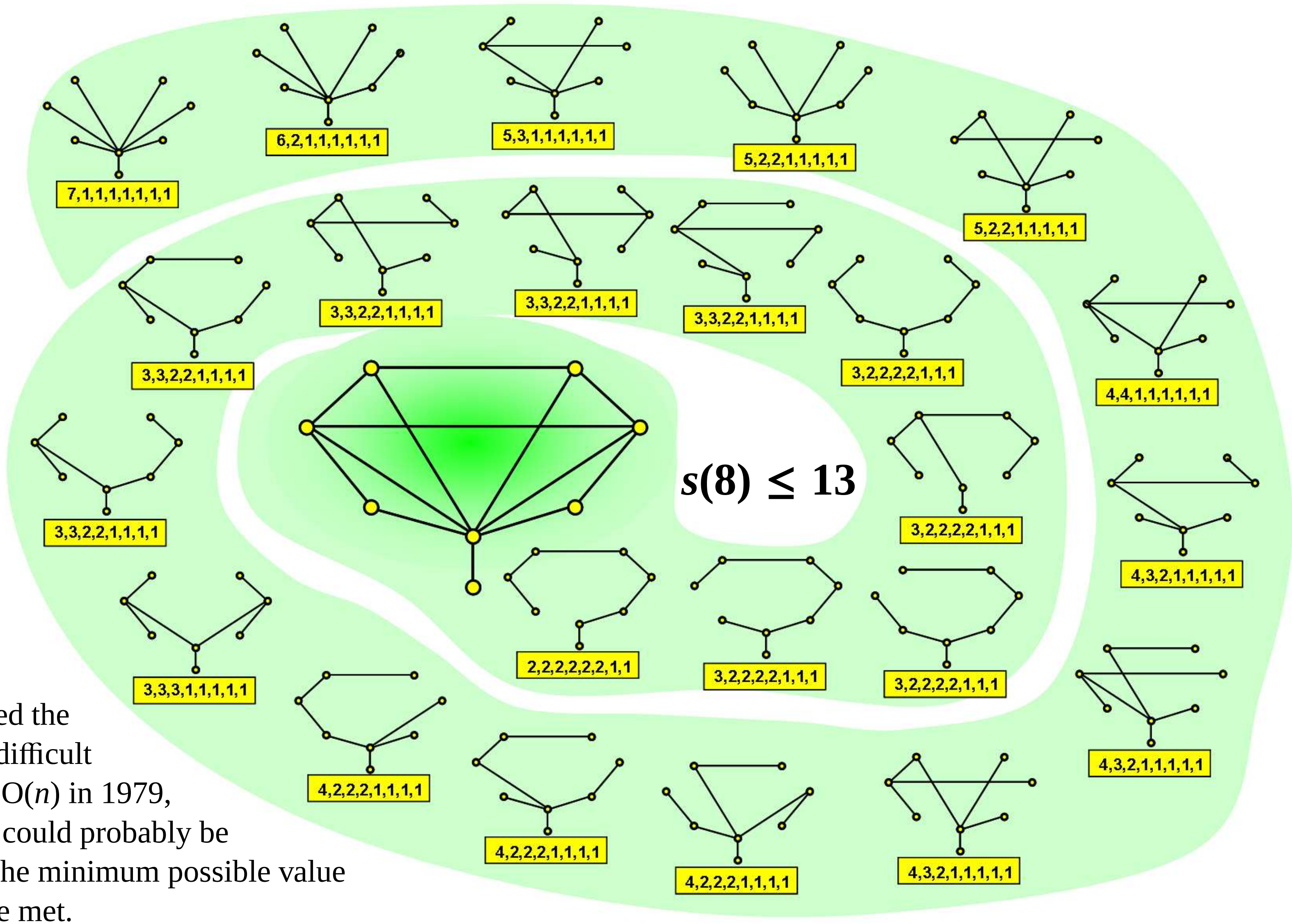
THEOREM OF THE DAY

The Panarboreal Formula *Denote by $s(n)$ the minimum number of edges a graph G on n vertices can have so that any tree on n vertices is isomorphic to some spanning tree of G. Then $s(n) \sim cn \log n$ where c is a constant satisfying $1/2 \le c \le 5/\log 4$.*

There are 23 unlabelled trees having 7 edges; they are shown on the right, lexicographically ordered by degree sequence, together with an 8-vertex graph with 13 edges in which each one may be found as a subgraph. No formula is known for the values of $s(n)$, but it is easy to establish that $s(n) \ge (1/2)(n-1)\log n$. For, given k, $1 \le k \le n$, we may always choose a tree in whose degree sequence the k-th entry $\ge (n-1)/k$ (for example, the 6th tree around the spiral on the right has 2nd entry $4 \ge (8-1)/2$). But now the same must hold for the degree sequence of any graph G containing this tree. So if G contains each n-vertex tree and has degree sequence, say, $(d_1, \ldots, d_n)$, then the number of edges in G is $= \frac{1}{2}\sum_{k=1}^{n} d_k \ge \frac{1}{2}\sum_{k=1}^{n}(n-1)/k > \frac{1}{2}(n-1)\log n$. So $s(n) > \frac{1}{2}n\log n - O(\log n) \sim \frac{1}{2}n\log n$.



Fan Chung and Ron Graham proved the easy lower bound on $s(n)$ and the difficult upper bound of $(5/\log 4)n\log n + O(n)$ in 1979, mentioning that $5/\log 4 \approx 3.6067$ could probably be improved, possibly even down to the minimum possible value of $1/2$. This challenge has yet to be met.

Web link: math.ucsd.edu/~fan/ an Aladdin's cave: *all* Chung's papers; the one concerned is "On universal graphs for spanning trees", *JLMS*, 1983.

Further reading: *Erdős on Graphs: His Legacy of Unsolved Problems*, by Fan Chung and Ronald Graham, AK Peters, 1998, section 3.5.1.