THEOREM OF THE DAY

The Panarboreal Formula Denote by $s(n)$ the minimum number of edges a graph $G$ on $n$ vertices can have so that any tree on $n$ vertices is isomorphic to some spanning tree of $G$. Then $s(n) \sim cn \log n$ where $c$ is a constant satisfying $1/2 \leq c \leq 5/\log 4$.

There are 23 unlabelled trees having 7 edges; they are shown on the right, lexicographically ordered by degree sequence, together with an 8-vertex graph with 13 edges in which each one may be found as a subgraph. No formula is known for the values of $s(n)$, but it is easy to establish that $s(n) \geq (1/2)(n - 1) \log n$. For, given $k$, $1 \leq k \leq n$, we may always choose a tree in whose degree sequence the $k$-th entry $\geq (n - 1)/k$ (for example, the 6th tree around the spiral on the right has 2nd entry $4 \geq (8 - 1)/2$).

But now the same must hold for the degree sequence of any graph $G$ containing this tree. So if $G$ contains each $n$-vertex tree and has degree sequence, say, $(d_1, \ldots, d_n)$, then the number of edges in $G$ is $\geq \frac{1}{2} \sum_{k=1}^{n} d_k \geq \frac{1}{2} \sum_{k=1}^{n} (n - 1)/k > \frac{1}{2}(n - 1) \log n$. So $s(n) > \frac{1}{2} n \log n - O(\log n) \sim \frac{1}{2} n \log n$.

Fan Chung and Ron Graham proved the easy lower bound on $s(n)$ and the difficult upper bound of $(5/ \log 4)n \log n + O(n)$ in 1979, mentioning that $5/ \log 4 \approx 3.6067$ could probably be improved, possibly even down to the minimum possible value of $1/2$. This challenge has yet to be met.