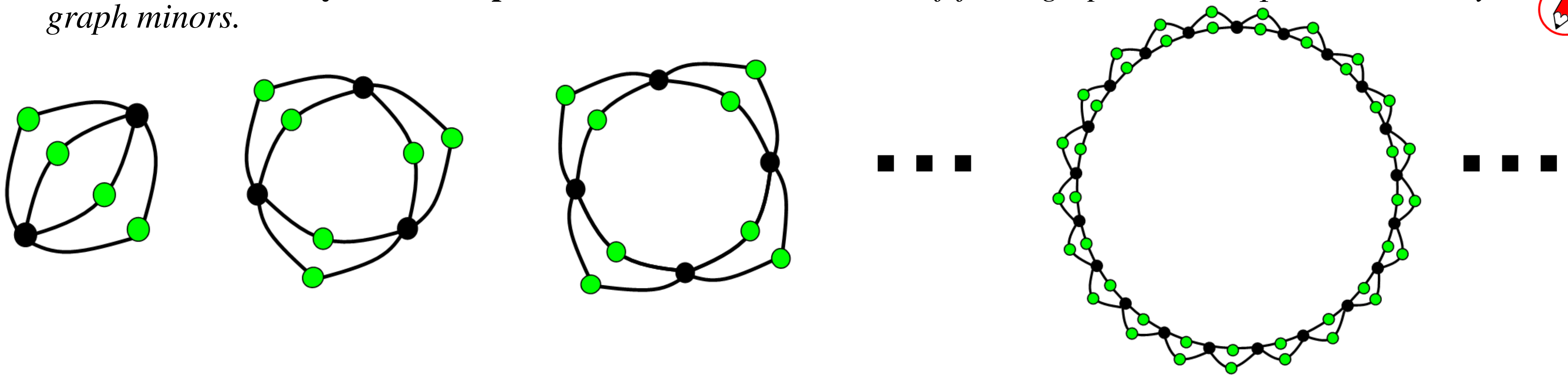
THEOREM OF THE DAY

The Robertson–Seymour Graph Minors Theorem *The set of finite graphs is well-quasi-ordered by graph minors.*



An order relation ≤ on a set is a *well-quasi-ordering* if there are no infinite antichains, that is to say, any infinite sequence $x_0, x_1, \ldots,$ contains some pair of elements x, y satisfying $x \leq y$. Here, we are interested in graphs G and H and in $H \leq G$ meaning H is a graph minor of G: H is a subgraph of G with some edges contracted.

Let P be a property of graphs (such as the property of being planar, say, or of having no cycles) which is *closed under graph minors*: G has property P and $H \leq G$ means that H also has property P. Now let $\mathcal{F}$ be a family of graphs failing to have property P. Suppose $\mathcal{F}$ is *minor minimal*: any minor of any graph in $\mathcal{F}$ *does* have P. Then, and this is the payoff, the confirmation of a famous conjecture of Wagner (see below), $\mathcal{F}$ must be finite. For, if not, list the graphs of $\mathcal{F}$: $x_0, x_1, \ldots$. By Robertson-Seymour, $\mathcal{F}$ contains graphs x and y with $x \leq y$. Then x has property P by the assumption that $\mathcal{F}$ was minor-minimal, but $x \in \mathcal{F}$ means that x does *not* have property P. This contradiction means that $\mathcal{F}$ cannot be infinite. QED.

The picture above addresses a technical question: why can we not use *topological minors*, à la Kuratowski's Theorem, instead of graph minors, à la Wagner's? The infinite sequence shown here is well-quasi-ordered by graph minors but not by topological minors: no graph can be obtained as a subgraph of any other by replacing edges with disjoint paths. So in the ordering by topological minors, this is an infinite antichain.

In 1937 Klaus Wagner, proving that graph planarity could be characterised using graph minors in place of topological minors (Kuratowski's theorem), conjectured that any such property characterisation could be achieved using a finite set of forbidden minors (for planarity, K_5 and $K_{3,3}$). Robertson and Seymour's 2004 proof marked a coming of age for graph theory as a profound branch of modern mathematics. It required hundreds of pages of dense mathematical reasoning spread over 20 lengthy journal papers; three quarters of the work lies in establishing a structural characterisation of minor-closed families of graphs; well-quasi ordering, and very much else, derives from this structure theorem.

Web link: www.ams.org/featurecolumn/archive/gminor.html. Modern developments and context are described in the following excellent survey: homepages.mcs.vuw.ac.nz/~whittle/pubs/preprint-structure-in-minor-closed-classes-of-matroids.pdf

Further reading: *Graph Theory (4th Edition)* by Reinhard Diestel, Springer New York, 2010, chapter 12.