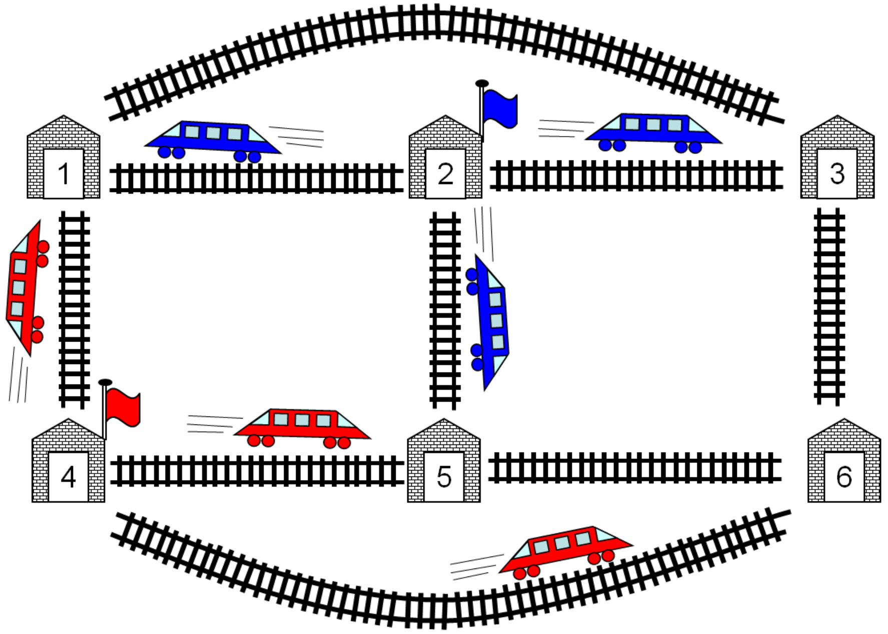
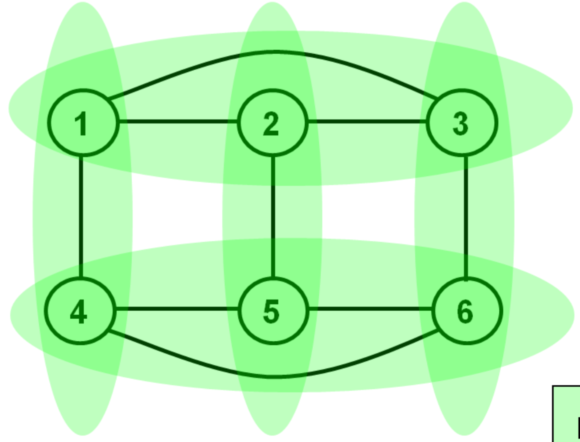
THEOREM OF THE DAY

The Strong Perfect Graph Theorem *A graph G is perfect if and only if neither G nor its complement $\overline{G}$ contains an induced odd circuit of length ≥ 5.*





Today's depot stocks

v_1	v_2	v_3	v_4	v_5	v_6
10	4	6	9	2	4

	v_1	v_2	v_3	v_4	v_5	v_6
c_1	1	1	1	0	0	0
c_2	1	0	0	1	0	0
c_3	0	1	0	0	1	0
c_4	0	0	1	0	0	1
c_5	0	0	0	1	1	1

Maximise: $\Sigma\, v_i \times (\text{stock } i)$,
subject to constraints:

$$
\begin{array}{lllllllll}
v_1 & + & v_2 + v_3 & & & & & \leq & 1 \\
v_1 & & & + & v_4 & & & \leq & 1 \\
& v_2 & & + & & v_5 & & \leq & 1 \\
& & v_3 & & + & & v_6 & \leq & 1 \\
& & & & v_4 + v_5 + v_6 & & & \leq & 1
\end{array}
$$

An induced odd circuit is an odd-length, circular sequence of edges having no 'short-circuit' edges across it, while $\overline{G}$ is the graph obtained by replacing edges in G by non-edges and vice-versa. A perfect graph G is one in which, for every induced subgraph H, the size of a largest *clique* (that is, maximal complete subgraph) is equal to the chromatic number of H (the least number of vertex colours guaranteeing no identically coloured adjacent vertices). This deep and subtle property is confirmed by today's theorem to have a surprisingly simple characterisation, whereby the railway above is clearly perfect.

The railway scenario illustrates just one way in which perfect graphs are important. We wish to dispatch goods every day from depots $v_1, v_2, \ldots$, choosing the best-stocked depots but subject to the constraint that we nominate at most one depot per network clique, so as to avoid head-on collisions. The depot-clique incidence relationship is modelled as a 0-1 matrix and we attempt to replicate our constraint numerically from this as a set of inequalities (far right, bottom). Now we may optimise dispatch as a standard *linear programming* problem **unless...** the optimum allocates a fractional amount to each depot, failing to respect the one-depot-per-clique constraint. A 1975 theorem of V. Chvátal asserts: *if a clique incidence matrix is the constraint matrix for a linear programme then an integer optimal solution is guaranteed if and only if the underlying network is a perfect graph.*

Exercise: find a choice of depots which is better than that depicted, where v_2 and v_4 have total stock of $4 + 9 = 13$.

Claude Berge's *Strong Perfect Graph Conjecture* defied experts for over 40 years, from 1960 until just before his death, in June 2002, aged 76. A 2001 proof for square-free graphs by Michele Conforti, Gérard Cornuéjols and Kristina Vušković was quickly followed by a general proof by Maria Chudnovsky and Paul Seymour, building on earlier work with Neil Robertson and Robin Thomas.

Web link: users.encs.concordia.ca/~chvatal/perfect/spgt.html

Further reading: *Graph Theory* by Reinhard Diestel, Springer, 2017.