THEOREM OF THE DAY

A Tripartite Turán Theorem Let $G$ be a tripartite graph with parts $A$, $B$ and $C$. Let $d(A, B)$ denote the density of edges between $A$ and $B$, i.e. $d(A, B) = (\text{no. of edges between } A \text{ and } B)/(|A||B|)$, and similarly for $d(A, C)$ and $d(B, C)$. Now denote $d(A, B)$, $d(B, C)$ and $d(A, C)$ by $\gamma$, $\alpha$ and $\beta$, respectively, and let $\phi$ denote the golden ratio. Then, if $\alpha, \beta, \gamma > 1/\phi$, $G$ has a triangle.

In 1907 the Dutch mathematician Willem Mantel published the solution to a problem posed by him in Wiskundige Opgaven (Mathematical Exercises): if a graph $G$ on $n$ vertices has $m$ edges then how large must $m$ be for $G$ necessarily to contain a triangle (a cyclic path of three edges, otherwise known as $K_3$)? The answer, now known as Mantel’s Theorem is: $m > \lfloor n^2/4 \rfloor$, and this is best possible, because a bipartite graph contains no triangles but may, as with the one shown above left, have exactly $\lfloor n^2/4 \rfloor$ edges. The critical edge density, then, is $1/2$: if more than $1/2$ of the $\left(\begin{array}{c} n \\ 2 \end{array}\right)$ possible edges are present then a triangle is inevitable. In the bipartite graph all the edge density occurs between the two parts of the partition; it is natural to ask, what are the edge densities, $\alpha$, $\beta$ and $\gamma$ between the three parts of a tripartite graph that will force a triangle? Adrian Bondy, Jian Shen, Stéphan Thomassé and Carsten Thomassen published the answer in 2006: a triangle is forced when:

$$\alpha\beta + \gamma > 1,$$

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$$\gamma\alpha + \beta > 1.$$  \hspace{1cm} (1)

Now the famous equation $\phi^2 - \phi - 1 = 0$ rearranges to give $(1/\phi)^2 + 1/\phi = 1$, so that $1/\phi$ supplies a simultaneous critical density for $\alpha$, $\beta$ and $\gamma$. Again, this is best possible: in the tripartite graph above right, the $n_A$ vertices of part $A$ are subdivided in proportion to $\beta$, the edge density from $C$; and all this density is incident with the ‘top’ $\beta n_A$ vertices of $A$. $B$ is similarly subdivided in proportion to $\alpha$. There is no density between the top vertices of $A$ and $B$ which makes triangles impossible, and we have $\alpha\beta + \gamma = \alpha\beta + (1 - \alpha\beta) = 1$, so one of the inequalities in (1) fails to hold.

Although extremal graph theorists trace their subject back to Mantel’s famous problem it is the 1941 generalisation from triangles $K_3$ to arbitrary complete graphs $K_r$ by Paul Turán that underlies modern work in the area.

Web link: vc.bridgew.edu/honors/proj/234/; “Extremal Graph Theory: Turán’s Theorem” by Vincent Vascimini