THEOREM OF THE DAY

Tutte's Golden Identity *Let T be an n-vertex planar triangulation with chromatic polynomial $P(T,\lambda)$, and let φ denote $\frac{1}{2}(1+\sqrt{5})$, the golden ratio. Then*

$$P(T,\varphi+2)=(\varphi+2)\varphi^{3n-10}(P(T,\varphi+1))^2.$$

A planar triangulation T is a graph embedded in the plane in such a way that every face is a triangle. Then $P(T,\lambda)$ is the wonderful polynomial whose value at any positive integer value of λ is the number of ways to use λ colours to properly colour the vertices of T, that is, with no adjacent vertices having the same colour. A 'random' 10-vertex planar triangulation T is shown on the right; its chromatic polynomial is $P(T,\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)\left(\lambda^6-18\lambda^5+141\lambda^4-617\lambda^3+1588\lambda^2-2265\lambda+1385\right)$, shown in the background, plotted between $\lambda=2$ and $\lambda=3.7$. It has zeros at 2 and 3 (circled) since no proper colouring is possible with fewer than 4 colours. A further zero occurs at almost exactly $\varphi+1\approx 2.618$; small values of P in this vicinity are guaranteed by a striking **Golden Inequality** for planar triangulations:

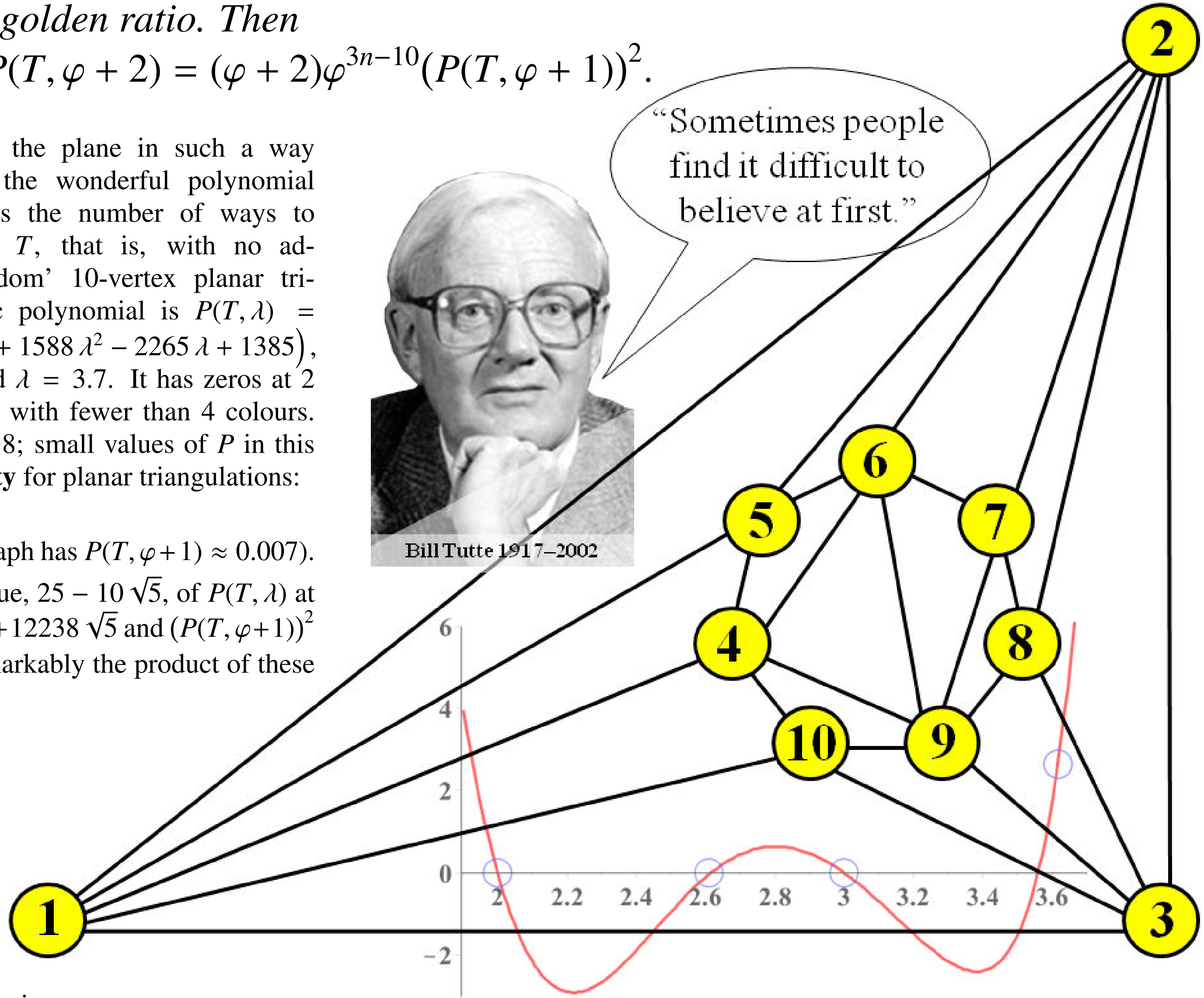
$$0<|P(T,\varphi+1)|\leq\varphi^{5-n},$$

(the right-hand side is about 0.09 for $n=10$ and our graph has $P(T,\varphi+1)\approx 0.007$).

The right-most circled point on our plot shows the value, $25-10\sqrt{5}$, of $P(T,\lambda)$ at $\lambda=\varphi+2$. Meanwhile, $(\varphi+2)\varphi^{3n-10}$ evaluates to $27365+12238\sqrt{5}$ and $(P(T,\varphi+1))^2$ evaluates to $259205-115920\sqrt{5}$, and indeed and remarkably the product of these two numbers is $25-10\sqrt{5}$.

The triangulation property is essential: if we, say, insert a vertex into the edge from vertex 1 to vertex 2 then the two incident triangular faces become squares, and the identity is found to fail.

A consequence of the identity, combined with the Golden Inequality, is that $P(T,\varphi+2)>0$. This was of interest in view of the proximity of $\varphi+2$ to 4: the Four Colour Theorem, eventually proved by other methods, asserts that $P(T,4)>0$ for all planar triangulations.



Chromatic polynomials were introduced into the study of 4-colourings in 1912 by George David Birkhoff. They were excessively laborious to compute before the advent of computers; catalogues were compiled in the 60s by Ruth Bari and by Dick Wick Hall and this led Bill Tutte and Gerald Berman to discover the Golden Inequality. Tutte subsequently discovered his Identity in 1970.

Weblink: www.maths.nottingham.ac.uk/personal/drw/PG/cp.hndt.pdf.

Further reading: Graph Theory As I Have Known It by William T. Tutte, OUP, 1998, Chapter 11. The above quote is on p. 134).