Tutte's Golden Identity *Let T be an n-vertex planar triangulation with chromatic polynomial $P(T,\lambda)$, and let φ denote $\frac{1}{2}(1+\sqrt{5})$, the golden ratio. Then*

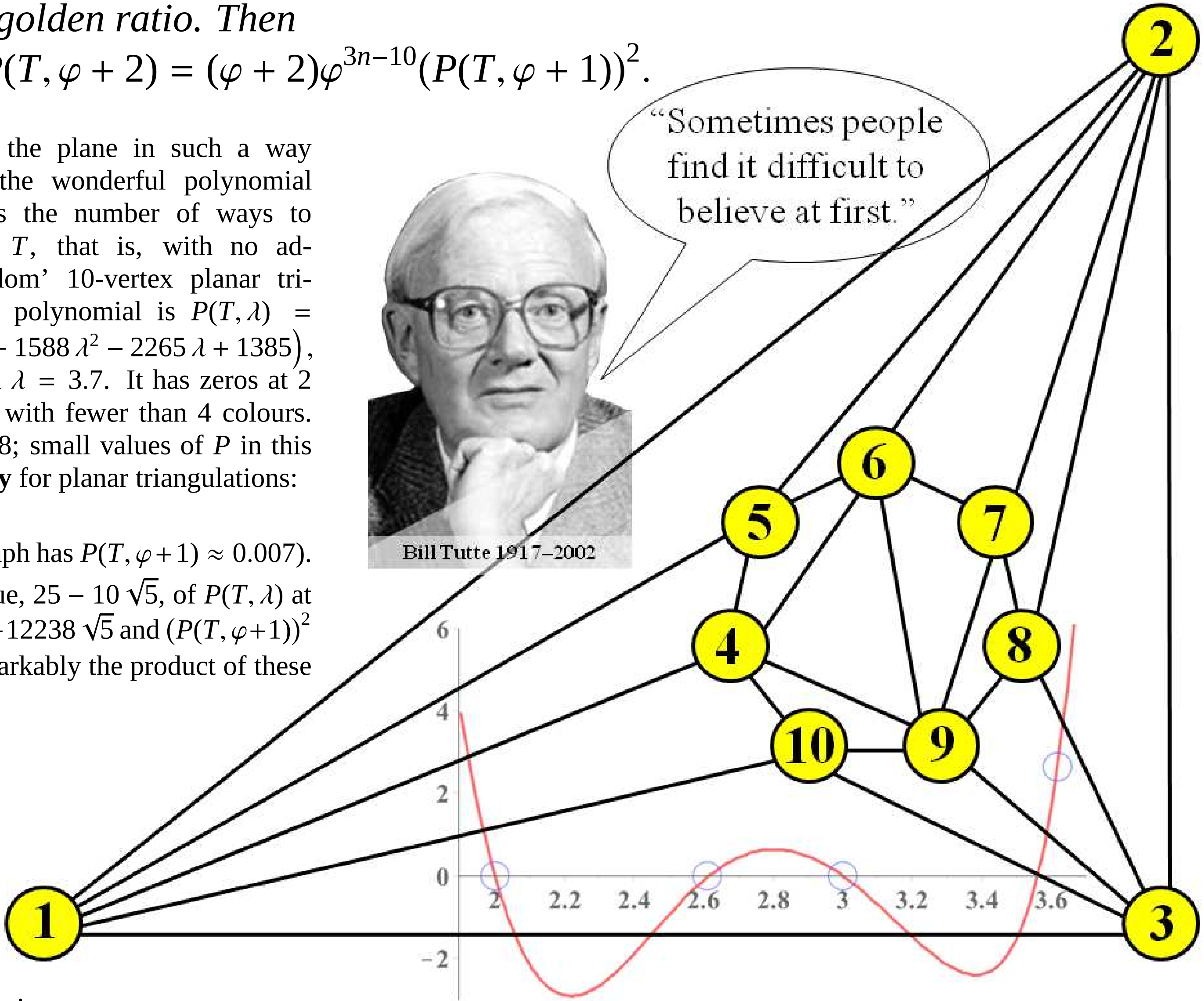
$$P(T,\varphi+2) = (\varphi+2)\varphi^{3n-10}(P(T,\varphi+1))^2.$$

A planar triangulation T is a graph embedded in the plane in such a way that every face is a triangle. Then $P(T,\lambda)$ is the wonderful polynomial whose value at any positive integer value of λ is the number of ways to use λ colours to properly colour the vertices of T, that is, with no adjacent vertices having the same colour. A 'random' 10-vertex planar triangulation T is shown on the right; its chromatic polynomial is $P(T,\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)\left(\lambda^6 - 18\lambda^5 + 141\lambda^4 - 617\lambda^3 + 1588\lambda^2 - 2265\lambda + 1385\right)$, shown in the background, plotted between $\lambda = 2$ and $\lambda = 3.7$. It has zeros at 2 and 3 (circled) since no proper colouring is possible with fewer than 4 colours. A further zero occurs at almost exactly $\varphi + 1 \approx 2.618$; small values of P in this vicinity are guaranteed by a striking **Golden Inequality** for planar triangulations:

$$0 < |P(T,\varphi+1)| \leq \varphi^{5-n},$$

(the right-hand side is about 0.09 for $n = 10$ and our graph has $P(T,\varphi+1) \approx 0.007$).

The right-most circled point on our plot shows the value, $25 - 10\sqrt{5}$, of $P(T,\lambda)$ at $\lambda = \varphi+2$. Meanwhile, $(\varphi+2)\varphi^{3n-10}$ evaluates to $27365+12238\sqrt{5}$ and $(P(T,\varphi+1))^2$ evaluates to $259205 - 115920\sqrt{5}$, and indeed and remarkably the product of these two numbers is $25 - 10\sqrt{5}$.



The triangulation property is essential: if we, say, insert a vertex into the edge from vertex 1 to vertex 2 then the two incident triangular faces become squares, and the identity is found to fail.

A consequence of the identity, combined with the Golden Inequality, is that $P(T,\varphi+2) > 0$. This was of interest in view of the proximity of $\varphi+2$ to 4: the Four Colour Theorem, eventually proved by other methods, asserts that $P(T,4) > 0$ for all planar triangulations.

Chromatic polynomials were introduced into the study of 4-colourings in 1912 by George David Birkhoff. They were excessively laborious to compute before the advent of computers; catalogues were compiled in the 60s by Ruth Bari and by Dick Wick Hall and this led Bill Tutte and Gerald Berman to spot a link to the golden ratio, subsequently formalised by Tutte in the above theorems.

Weblink: www.maths.nottingham.ac.uk/personal/drw/PG/cp.hndt.pdf.

Further reading: Graph Theory As I Have Known It by William T. Tutte, OUP, 1998, Chapter 11. The above quote is on p. 134).