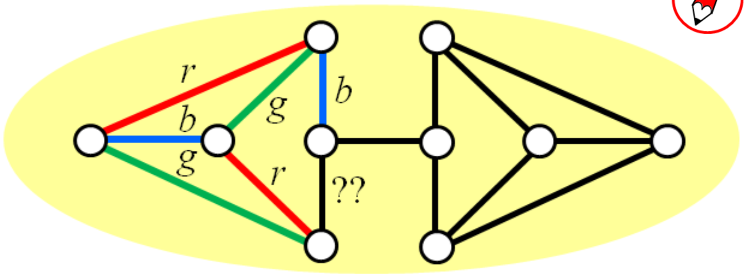
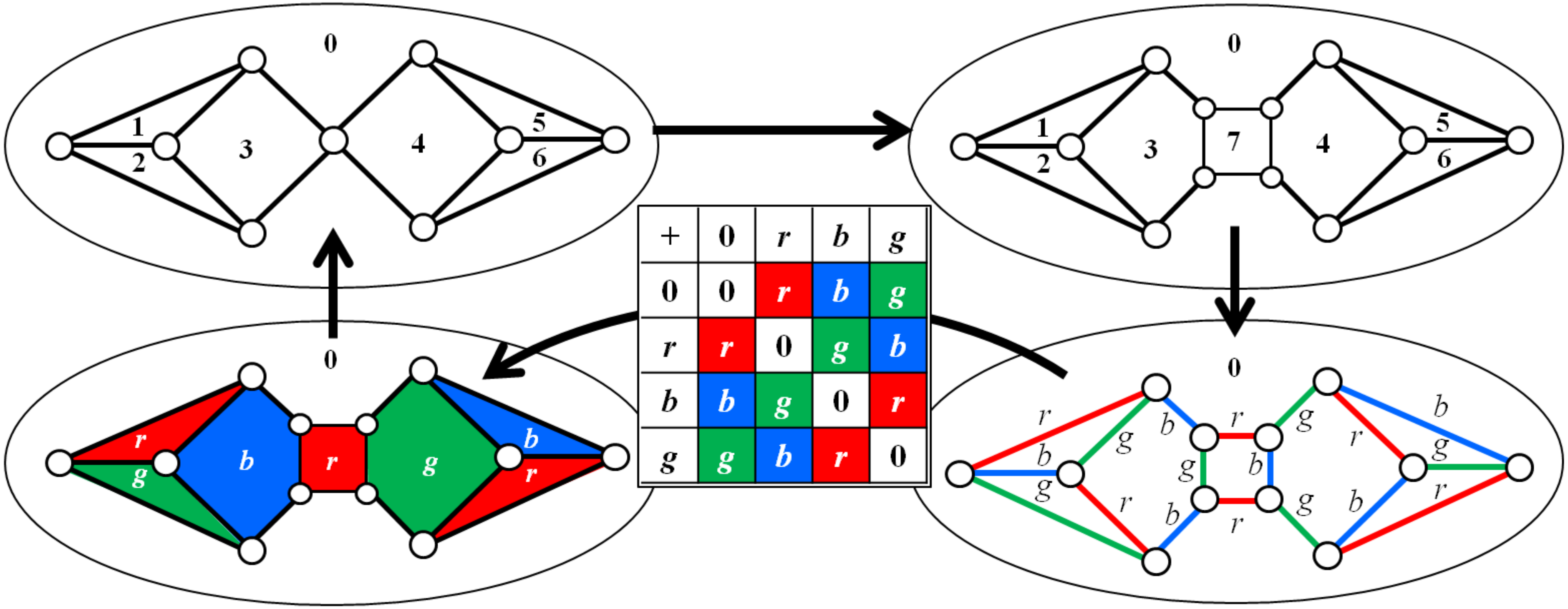
THEOREM OF THE DAY

Vizing's Theorem *A simple graph of maximum degree Δ has chromatic index Δ or $\Delta + 1$.*

An *edge colouring* assigns a colour to each edge of a graph G in such a way that no incident edges are assigned the same colour. The *chromatic index* of G, usually denoted by $\chi'(G)$ is the minimum number of colours for which an edge colouring is possible. This cannot be less than the greatest number of edges incident with a vertex: the *maximum degree*, denoted by Δ: all such edges must be coloured differently. Surprisingly, provided the graph is *simple* — no multiple edges or loops — at most one more colour is required. An example is shown on the right: this is a *3-regular* graph: all vertices have degree 3. So $\Delta = 3$ but, having coloured six edges using {red, blue, green} we are forced to add a fourth colour for the edge marked '??'. The graph is accordingly said to belong to *class 2*: requiring $\Delta + 1$ colours, as opposed to *class 1*: requiring the minimum possible Δ.



In fact our example graph was consigned to class 2 purely because it has a *bridge*: an edge whose deletion would separate the graph into two component pieces. Indeed, the study of edge colourings was inspired by Peter Guthrie Tait's 1880 observation that the Four Colour Theorem was equivalent to the assertion: *all bridgeless, 3-regular, planar graphs belong to class 1*.



+	0	r	b	g
0	0	r	b	g
r	r	0	g	b
b	b	g	0	r
g	g	b	r	0

The sequence above illustrates Tait's idea: the graph top left is planar: no edges cross and the drawing unambiguously partitions the plane into *faces*, including an outside face. Although not 3-regular it can be made so by the trick of replacing vertices of degree > 3 by cycles (the version on the right). Next, 3-colour the edges in accordance with Tait's assertion (bottom right). And then, starting with the outside face coloured '0', repeatedly cross edges into uncoloured faces, and colour these faces using the 'arithmetic table' shown in the middle. For example, crossing from the outside face into the left-hand pentagon colours it $0 + b = b$; continuing right into the square colours this $b + g = r$. The result is a 4-colouring of the faces, which is undisturbed on shrinking artificially created cycles back to single vertices.

This theorem was discovered by Vadim G. Vizing in 1964 and, independently, by Ram Prakash Gupta in 1966.

Weblink: An elegant double-induction proof: homepages.cwi.nl/~lex/files/vizing.pdf; and an interview with Vizing: www.ems-ph.org/journals/newsletter/pdf/2000-12-38.pdf.

Further reading: *Introduction to Graph Theory, 5th Ed.,* by Robin J. Wilson, Prentice Hall, 2010.