Wagner’s Theorem A graph $G$ is planar if and only if it contains neither $K_5$ nor $K_{3,3}$ as a graph minor.

The graph above left is the famous Petersen graph. Its nonplanarity can be confirmed by Kuratowski’s theorem: $K_{3,3}$ is a topological minor. Wagner showed that the theorem still works when topological minors are replaced by graph minors. $H$ is a graph minor of $G$ if $G$ has a subgraph which can be reduced to $H$ by contracting edges (i.e. shrinking them until their end-vertices coincide). In the picture $K_5$, above centre, can be seen, by contracting the dotted edges, to be a graph minor of the Petersen graph. The same is true of $K_{3,3}$, above right: it is obtained by contracting edges in the subgraph resulting from a single edge deletion. So both $K_5$ and $K_{3,3}$ are graph minors of the Petersen graph (whereas $K_5$ is not, in fact, a topological minor).

Klaus Wagner’s alternative formulation of the ‘forbidden substructure’ characterisation of planarity appeared in 1937, seven years after Kuratowski’s. The difference between graph minors and topological minors may appear subtle but its consequences are profound: Wagner ventured further to make the enormously bold conjecture, which is false for topological minors, that for any property characterised by graph minors the set of minors could be taken to be finite. The proof, completed nearly 70 years later, is the centrepiece of a whole branch of combinatorics, known colloquially as ‘Robertson-Seymour’ theory.

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