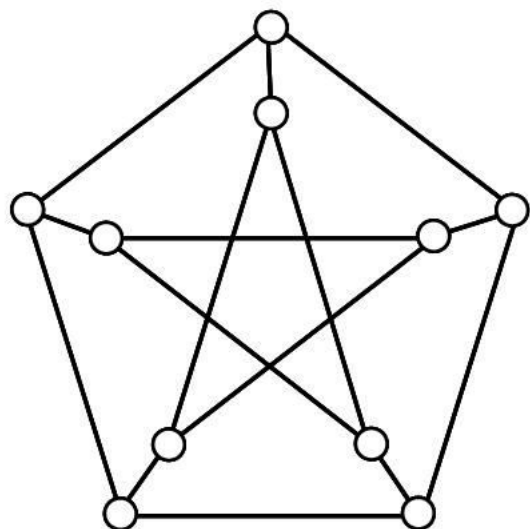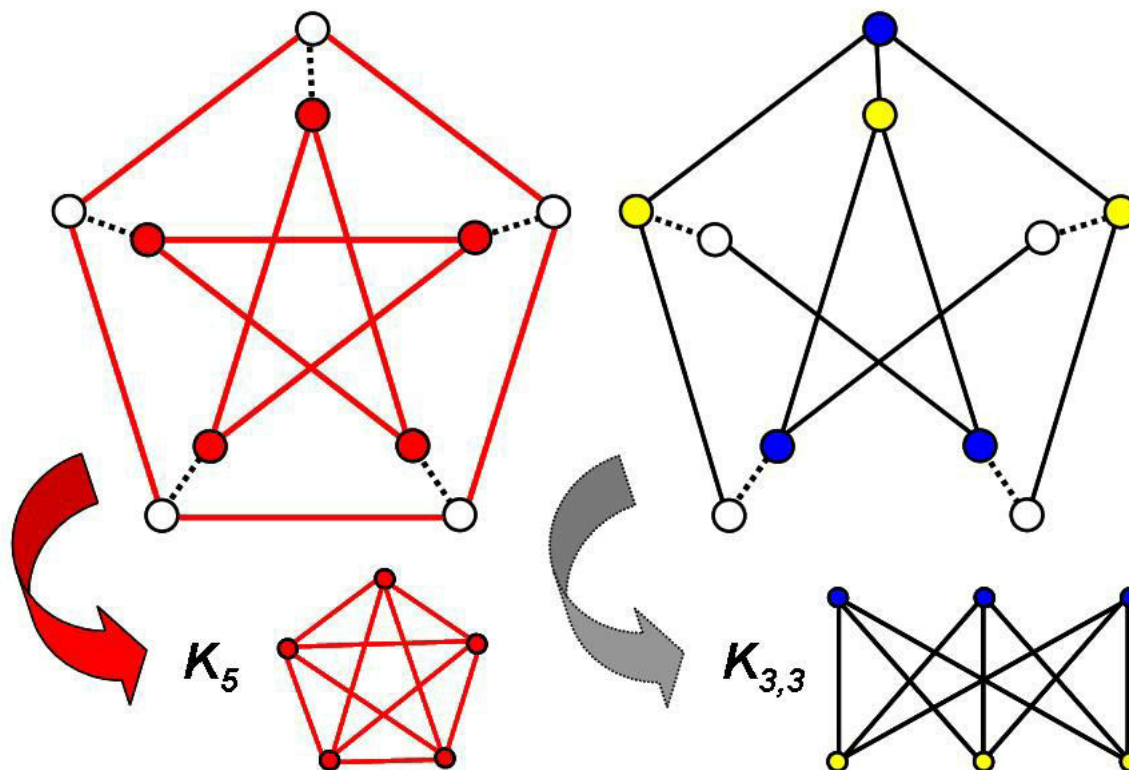# THEOREM OF THE DAY

Wagner's Theorem *A graph G is planar if and only if it contains neither K_5 nor $K_{3,3}$ as a graph minor.*

The Petersen Graph

K_5

$K_{3,3}$

The graph above left is the famous *Petersen graph*. Its nonplanarity can be confirmed by Kuratowski's theorem: $K_{3,3}$ is a topological minor. Wagner showed that the theorem still works when topological minors are replaced by *graph minors*. *H* is a graph minor of *G* if *G* has a subgraph which can be reduced to *H* by *contracting* edges (i.e. shrinking them until their end-vertices coincide). In the picture K_5, above centre, can be seen, by contracting the dotted edges, to be a graph minor of the Petersen graph. The same is true of $K_{3,3}$, above right: it is obtained by contracting edges in the subgraph resulting from a single edge deletion. So both K_5 and $K_{3,3}$ are graph minors of the Petersen graph (whereas K_5 is not, in fact, a topological minor).

Klaus Wagner's alternative formulation of the 'forbidden substructure' characterisation of planarity appeared in 1937, seven years after Kuratowski's. The difference between graph minors and topological minors may appear subtle but its consequences are profound: Wagner ventured further to make the enormously bold conjecture, which is false for topological minors, that for *any* property characterised by graph minors the set of minors could be taken to be *finite*. The proof, completed nearly 70 years later, is the centrepiece of a whole branch of combinatorics, known colloquially as 'Robertson-Seymour' theory.

Web link: www.ams.org/featurecolumn/archive/gminor.html

Further reading: *Introduction to Graph Theory, 5th Ed.,* by Robin Wilson, Prentice Hall, 2010.