Notes for Science and Engineering Foundation Maths 1 and Maths 2

by

Robin Whitty BSc PhD CMath FIMA

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Contents

Preface

These notes are intended to put on record what is presented during SEFP Maths1 lectures (year 2009/2010). Each week's lectures will appear as a new chapter, although not until at least a week after the lectures have taken place.

The two most important things to look out for are

Numbered Equations These are the rules and formulae which you are supposed to know.

Worked Examples These are similar in content to what you will meet in the marked assignment sheets and in the exams. The level of the Worked Examples is supposed to be *at least as hard* as the assignment and exam questions. If you study them well and feel confident you understand them then you should have no trouble succeeding on this course.

These notes are not intended to provide supplementary material; the course book *Core Maths for Advanced Level, 3rd ed.* by L. Bostock F.S. Chandler, Nelson Thornes, 2000 is very comprehensive and contains a large number of exercises with solutions. Exercise sets which are particularly relevant are provided in the Maths1 Teaching Scheme which is on the web at:
http://webspace.qmul.ac.uk/rwwhitty/SEFP/Maths1/Maths1.html.

The weekly tutorial sheets for the course are the best source of practice material because you can follow up your practice by talking to a course tutor. Learning maths is like learning a foreign language: you have to use it to get good at it. Memorising formulae is like memorising tables of verbs: it is not a bad thing to do but it will not help you when faced with a practical situation, only practice will do that.

There are bound to be errors in this text, so if something looks wrong, it probably is! Or perhaps you think something could be explained better? Get you name in the list of acknowledgements below by emailing me about it at whittyr@lsbu.ac.uk.

These notes have benefited from feedback from the following people: A.N.Other.

Robin Whitty, October 2009

Chapter 2

Algebra 1

2.1 Quadratics

A *quadratic (polynomial)* is an expression of the form

$$ax^2 + bx + c, \quad a, b, c \in \mathbb{R} \tag{2.1}$$

a, *b* are called the *coefficients* of x^2 and x, respectively, and c is called the *constant term* (and it is the coefficient of x^0) and (the notation '$\in \mathbb{R}$' comes from set theory and is short-hand for "in the set of real numbers").

If we plot the values of a quadratic for different values of x it becomes a *function* $f(x) = ax^2 + bx + c$ called a *quadratic curve* as shown in figure 2.1.

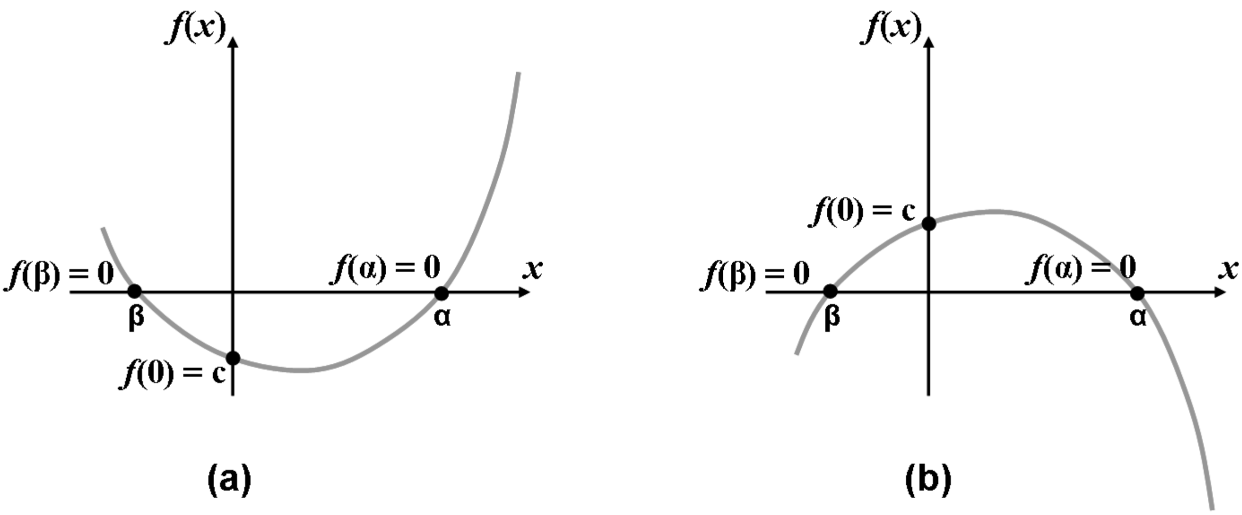

Figure 2.1: quadratic curves $f(x) = ax^2 + bx + c$ with (a) $a > 0, c < 0$ and (b) $a < 0, c > 0$.

Notice that both curves in figure 2.1 distinguish three points: when $x = 0$, the value of $f(x)$ is c and this is where the curve crosses the vertical axis; at $x = \alpha$ and $x = \beta$ the value of $f(x)$ is zero and this is where the curve crosses the horizontal axis. Because $f(x) = 0$ at α and β these x values are called *zeros* of the function $f(x)$.

A quadratic curve need not have any zeros: if the value of c in figure 2.1(a) were to lie above the x axis then the curve would not descend low enough to cross the x axis. The possible situations are sketched in figure 2.2

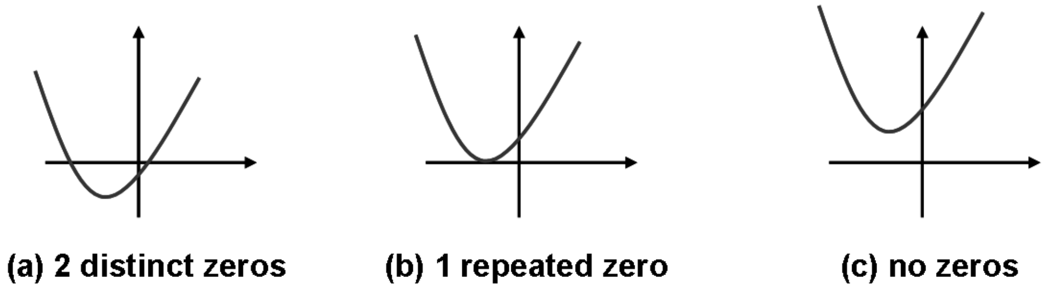

Figure 2.2: different possibilities for the zeros of a quadratic curve $f(x) = ax^2 + bx + c$, $a > 0$.

In the situation of figure 2.2(a) there are two zeros α and β and there is an infinite family of quadratic curves $f(x) = ax^2+bx+c$ which have these zeros, all of which are multiples of the other. They are also called *roots* of the quadratic polynomial $ax^2 + bx + c$; and they are also called *solutions* of the equation $ax^2 + bx + c = 0$.

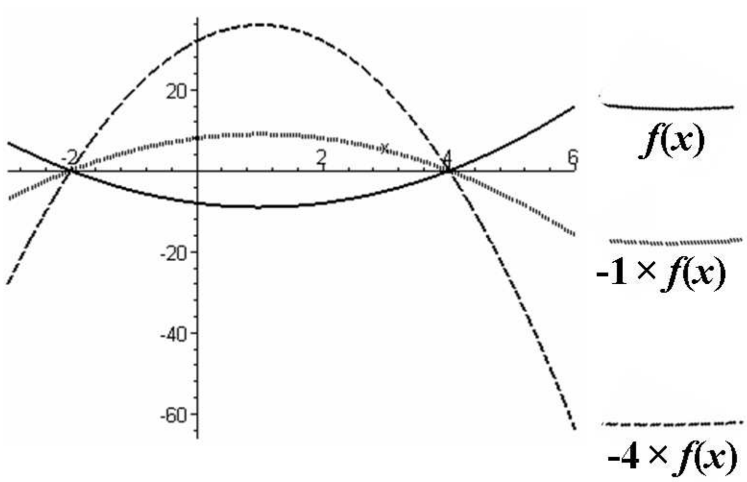

Figure 2.3: the quadratic curve $f(x) = x^2 - 2x - 8$ and two multiples of it, all having zeros at $x = 4$ and $x = -2$.

To summarise, we have

zeros of the curve $f(x) = ax^2 + bx + c$, the x values at which the curve intersects the x axis;

roots of the quadratic ax^2+bx+c, the values of x for which the quadratic evaluates to zero;

solutions of the equation $ax^2 + bx + c = 0$, the x values for which the equation is satisfied.

Mathematicians are not very careful about using these terms consistently but it is important to recognise that they refer to different things, even though *the values of x are the same in each case.*

Different multiples of a quadratic have the same roots, as seen in figure 2.3. Another way to view this is via *factorisation*. If the quadratic ax^2+bx+c has roots α and β then it can be *factorised* as $a(x-\alpha)(x-\beta)$. The two factors of this product, $(x-\alpha)$ and $(x-\beta)$ are called *linear factors*—they are linear because the highest power of x is 1. The quadratic $-3a^2-3bx-3c$ is factorised as $-3a(x-\alpha)(x-\beta)$. It has the same roots; the corresponding equations $a(x-\alpha)(x-\beta) = 0$ and $-3a(x-\alpha)(x-\beta) = 0$ have the same solutions.

The quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{2.2}$$

gives the zeros of

$$f(x) = \text{any constant} \times (a^x + bx + c).$$

Worked Example 6 *Factorise the quadratics (i)* $3x^2-2x-8$, *(ii)* $-9x^2+6x+24$

Solution Applying the quadratic formula in (i), with $a = 3$, $b = -2$ and $c = -8$:

$$x = \frac{2 \pm \sqrt{2 - 4 \times 3 \times (-8)}}{6} = \frac{1 \pm \sqrt{25}}{3} = 2 \text{ or } -\frac{4}{3}.$$

The linear factors are therefore $(x-2)$ and $(x-(-4/3)) = (x+4/3)$. The factorisation also has a multiple of $a = 3$:

$$3x^2 - 2x - 8 = 3(x-2)(x+4/3) = (3x-6)(x+4/3), \quad \text{or} \quad (x-2)(3x+4).$$

We observe that the quadratic in (ii) is just $-3 \times (3x^2 - 2x - 8)$ so this will have the same roots as in (i) but with a different multiplying factor:

$$-9x^2 + 6x + 24 = -3 \times (x-2)(3x+4) = (6-3x)(3x+4).$$

Given two quadratics, $ax^2 + bx + c$ and $px^2 + qx + r$, we write

$$ax^2 + bx + c = px^2 + qx + r \tag{2.3}$$

if they have the same value for every value of x. This is true if and only if $a = p$ and $b = q$ and $q = r$. This is called *equating coefficients*. Sometimes given a quadratic $ax^2 + bx + c$ we may use its roots to define a new quadratic $px^2 + qx + r$ which we can determine, without knowing these roots, by equating coefficients.

Worked Example 7 *Suppose the roots of* $3x^2 + 2x - 1$ *are* α *and* β. *Suppose another quadratic,* $px^2 + qx + r$, *has roots* $1/\alpha$ *and* $1/\beta$. *If* $p = 1$, *what are* q *and* r*?*

Solution We are told that $3x^2 + 2x - 1$ factorises with linear factors $(x - \alpha)$ and $(x-\beta)$. Since the coefficient of x^2 is 3, the factorisation is $3(x-\alpha)(x-\beta)$. Expanding the brackets and equating coefficients, we get

$$3(x - \alpha)(x - \beta) = 3x^2 - 3(\alpha + beta) + 3\alpha\beta = 3x^2 + 2x - 1,$$

so $\alpha + \beta = -2/3$ and $\alpha\beta = -1/3$. The second quadratic $x^2 + qx + r$ has linear factors $(x - 1/\alpha)$ and $(x - 1/\beta)$, so we have:

$$\begin{aligned} (x - \frac{1}{\alpha})(x - \frac{1}{\beta}) &= x^2 - (\frac{1}{\alpha} + \frac{1}{\beta}) + \frac{1}{\alpha\beta} \\ &= 3x^2 - \frac{\alpha + \beta}{\alpha\beta} x + \frac{1}{\alpha}\beta. \end{aligned}$$

2.2 Polynomial Division and the Remainder Theorem

We now consider polynomials with powers of x higher than two. The highest power of x in a polynomial is called the *degree* of the polynomial. Up to x^5, degree 5, the polynomials, and the corresponding curves (with the polynomial regarded as a function $f(x)$) and equations (with the polynomial equated to zero, $f(x) = 0$), are given names for convenience and some typical curves are shown in figure 2.4.

Consider:

$$\frac{21}{7} = \frac{3 \times 7 + 4}{7} = \frac{3 \times 7}{7} + \frac{4}{7} = 3 + \frac{4}{7}.$$

We say that when 7 divides into 21 the *quotient* is 3 (the number of times 7 "goes into" 21) and the *remainder* is 4 (what is "left over").

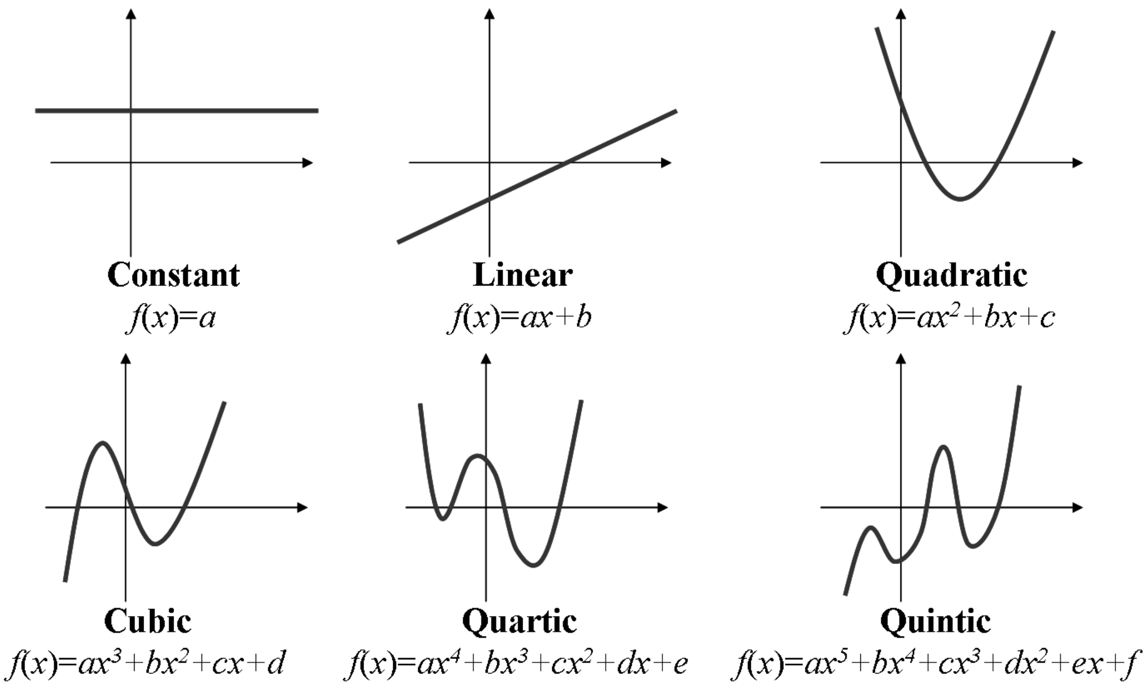

Figure 2.4: curves corresponding to the polynomials of degree zero up to 5.

We can divide one polynomial into another in a manner analogous to division of real numbers. A procedure similar to long division is applied:

Worked Example 8 *Divide the cubic polynomial* $2x^3 + x^2 + 4$ *by* $x - 1$.

Solution We find the quotient by repeatedly multiplying $x - 1$ by the power of x needed to give each power of x in the dividend $2x^3 + x^2 + 4$. We must be careful to include even those powers of x which are missing (i.e. have coefficient zero):

		$2x^2$	$+$	$3x$	$+$	3			$\longleftarrow$ quotient
	$(x-1)$	$2x^3$	$+$	x^2	$+$	$0x$	$+$	4	
$2x^2 \times (x-1)$	$\longrightarrow$	$2x^3$	$-$	$2x^2$					
subtract:				$3x^2$					
$3x \times (x-1)$	$\longrightarrow$			$3x^2$	$-$	$3x$			
subtract:						$3x$			
$3 \times (x-1)$	$\longrightarrow$					$3x$	$-$	3	
subtract:								7	$\longleftarrow$ remainder

So

$$\frac{2x^3 + x^2 + 4}{x-1} = \frac{(2x^2 + 3x + 3) \times (x-1) + 7}{x-1} = \frac{(2x^2 + 3x + 3) \times (x-1)}{x-1} + \frac{7}{x-1}$$

$$= 2x^2 + 3x + 3 + \frac{7}{x-1}.$$

In general, as with example 8, if $P(x)$ and $P'(x)$ are polynomial functions of x then dividing P by P' gives:

$$\frac{P(x)}{P'(x)} = \text{quotient } + \frac{\text{remainder}}{P'(x)} \quad \text{or} \tag{2.4}$$

$$P(x) = \text{quotient } \times P'(x) + \text{remainder} \tag{2.5}$$

We write $P(x)$ and $P'(x)$ rather than $f(x)$ and $f'(x)$ to emphasise the fact that we are talking about polynomials rather than functions or curves. If we *do* regard $P'(x)$ as a function then it may have a zero, a point $x = \alpha$ with $P'(\alpha) = 0$. What happens in (2.5) if we put $x = \alpha$? We get the following, which is true for polynomials of any degree:

Theorem 9 *Suppose a polynomial $P(x)$ when divided by a polynomial $P'(x)$ has remainder $R(x)$. Suppose $P'(x)$ has a zero at $x = \alpha$. Then $R(\alpha) = P(\alpha)$. In particular, if $P(x)$ is divided by $x - \alpha$ then the remainder is $P(\alpha)$.*

In example 8 we found that dividing $2x^3 + x^2 + 4$ by $x - 1$ gave remainder 7. So the Remainder Theorem tells us that $2 \times 1^3 + 1^2 + 4$ must have value 7, as indeed it does.

Proof of Theorem 9. From (2.5),

$$\begin{aligned} P(x) &= \text{quotient } \times P'(x) + R(x) \\ \text{So} \quad P(\alpha) &= \text{quotient } \times P'(\alpha) + R(\alpha) \\ &= \text{quotient } \times 0 + R(\alpha) \\ &= R(\alpha). \end{aligned}$$

□

(The little box is the way mathematicians often mark the end of proofs. You do not have to remember this proof and we shall not normally write them out, but you should note that every theorem has to have a proof; proofs are what allow engineers and scientists to use theorems with confidence when they build passenger aircraft or nuclear reactors or whatever.)

Worked Example 10 *Evaluate $P(x) = x^6 - 103x^5 + 396x^4 + 3x^2 - 296x - 101$ at $x = 99$.*

Solution This is not a calculation you can do directly, even with a calculator (99^6 is a number 12 digits long!). However, the Remainder Theorem says that $P(99)$ is the remainder when $P(x)$ is divided by $(x - 99)$. So we divide:

$$
\begin{array}{r|ccccccccccccc}
 & x^5 & - & 4x^4 & & & & & + & 3x & + & 1 & & \\
\cline{2-14}
(x-99) & x^6 & - & 103x^5 & + & 396x^4 & + & 0x^3 & + & 3x^2 & - & 296x & - & 101 \\
 & x^6 & - & 99x^5 & & & & & & & & & & \\
 & & - & 4x^5 & & & & & & & & & & \\
 & & - & 4x^2 & + & 396x^4 & & & & & & & & \\
 & & & & & 0 & & & & & & & & \\
 & & & & & & & & & 3x^2 & - & 297x & & \\
 & & & & & & & & & & & x & & \\
 & & & & & & & & & & & x & - & 9 \\
 & & & & & & & & & & & & & -2 \\
\end{array}
$$

The remainder is -2, so $P(99) = -2$.

Worked Example 11 *Suppose we have a polynomial $P(x) = 3x^2 + px + q$. You are told that $P(x)$ has remainder 3 on division by $x + 2$. You are also told that the roots of $P(x)$ are α and β and that they satisfy*

$$\alpha + \beta + \alpha\beta = -4.$$

Find p and q.

Solution We have

$$
\begin{aligned}
P(x) &= 3(x-\alpha)(x-\beta) \\
&= 3x^2 - 3(\alpha+\beta)x + 3\alpha\beta \qquad (1)
\end{aligned}
$$

By the Remainder Theorem, $P(-2) = 3$ because we are told that dividing $P(x)$ by $x - (-2)$ gives remainder 3. Substituting for x in (1):

$$
\begin{aligned}
3(-2)^2 - 3(\alpha+\beta)(-2) + 3\alpha\beta &= 3 \\
\text{or}\quad 6(\alpha+\beta) + 3\alpha\beta &= -9 \\
\text{or}\quad 2(\alpha+\beta) + \alpha\beta &= -3 \qquad (2) \\
\text{We we also told that} \\
\alpha + \beta + \alpha\beta &= -4 \qquad (3)
\end{aligned}
$$

Now (2) − (3) gives $\alpha + \beta = 1$ and substituting in (3):

$$
\begin{aligned}
1 + \alpha\beta &= -4 \\
\text{so}\quad \alpha\beta &= -5
\end{aligned}
$$

Since $P(x) = 3x^2 + px + q$ we equate coefficients with (1) and see that $p = -3(\alpha+\beta) = -3$ and $q = 3\alpha\beta = -15$.

2.3 Partial Fractions

We are given a polynomial division of the form $P(x)/P'(x)$, where $P'(x)$ is factorised like

$$P'(x) = \text{factor}_1 \times \text{factor}_2 \times \text{factor}_3 \times \ldots,$$

we try to rewrite the division as

$$\frac{P(x)}{P'(x)} = \frac{?}{\text{factor}_1} + \frac{?}{\text{factor}_2} + \frac{?}{\text{factor}_3} + \ldots.$$

We have to find out what the '?'s are to make this work. This is called a *partial fractions expansion* and is useful in integration and also in separating the different parts of a system which are undefined at certain values ($1/(x - \alpha)$ goes to infinity as x gets close to α and this may represent dangerous behaviour in an engineering system).

There is a basic rule of fractions that needs to be applied:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd} \tag{2.6}$$

(multiply both terms on the left-hand side by bd/bd and cancel to get the right-hand side).

Worked Example 12 *Show that*

$$\frac{19 - 8x}{3(2x-1)(x+2)} = \frac{2}{2x-1} - \frac{7}{3(x+2)}.$$

Solution Here the '?'s are given—you just have to check they are correct. Multiply top and bottom of the right-hand side by the product of the denominators and cancel where possible:

$$\begin{aligned} \frac{(2x-1) \times 3(x+2)}{(2x-1) \times 3(x+2)} \times \frac{2}{2x-1} - \frac{7}{3(x+2)} &= \frac{3(x+2) \times 2 - (2x-1) \times 7}{3(2x-1)(x+2)} \\ &= \frac{19 - 8x}{3(2x-1)(x+2)}, \end{aligned}$$

after rearranging terms.

If the highest power of x in $P(x)$ is lower than the highest power of x in $P'(x)$ then the fraction $P(x)/P'(x)$ is called *proper*. Otherwise the fraction is *improper* and we must divide $P'(x)$ into $P(x)$ to get an expression like equation (2.4) in the last section and then get a partial fractions expansion of the remainder part.

2.3.1 Proper Fractions

The procedure to follow depends on what is in the denominator of your fraction:

Linear factor: for each linear factor $\dfrac{\cdots}{(ax+b)}$ add $\dfrac{A}{ax+b}$ to the expansion;

Repeated factor: for each repeated linear factor $\dfrac{\cdots}{(ax+b)^2}$ add $\dfrac{A}{ax+b}+\dfrac{B}{(ax+b)^2}$ to the expansion;

Quadratic factor: for each quadratic factor $\dfrac{\cdots}{(ax^2+bx+c)}$ add $\dfrac{Ax+B}{ax^2+bx+c}$ to the expansion.

Worked Example 13 *Separate* $\dfrac{x+4}{(x+1)(x-2)(x-3)}$ *into partial fractions.*

Solution All the factors are linear and none are repeated, so our solution will be:

$$\frac{x+4}{(x+1)(x-2)(x-3)} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x-3}. \quad (1)$$

As in example 12 we multiply top and bottom of each term by the product of the factors and cancel. If we do this on both sides of (1) we get:

$$x+4 = (x-2)(x-3)A + (x+1)(x-3)B + (x+1)(x-2)C.$$

The idea is now to evaluate at suitable x values

$\boldsymbol{x=2}$: $2+4 = 0 + (3\times(-1))B + 0$

so $B = 6/-3 = -2$

$\boldsymbol{x=3}$: $3+4 = 0 + 0 + 4\times 1\times C$

so $C = 7/4$

$\boldsymbol{x=-1}$: $-1+4 = (-3)(-4)A + 0 + 0$

so $A = 1/4$

Solution: $$\frac{x+4}{(x+1)(x-2)(x-3)} = \frac{1}{4(x+1)} - \frac{2}{x-2} + \frac{7}{4(x-3)}.$$

Worked Example 14 *Separate* $\dfrac{x^2+1}{(x+1)(x-2)^2}$ *into partial fractions.*

Solution All the factors are linear but the second is repeated, so our solution will be:

$$\frac{x^2+1}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}. \quad (1)$$

Multiply top and bottom of both sides of (1) by the denominator of the right-hand-side to get:

$$x^2+1 = (x-2)^2A + (x+1)(x-2)B + (x+1)C.$$

Evaluate at suitable x values

$$\begin{array}{rcccccc}
\mathbf{x=2}: & 4+1 & = & 0 & + & 0 & + & 3C
\end{array}$$

so $C = 5/3$

$$\begin{array}{rcccccc}
\mathbf{x=-1}: & 1+1 & = & 9A & + & 0 & + & 0
\end{array}$$

so $A = 2/9$

There are no more denominator roots, so choose any 'easy' value:

$$\begin{array}{rcccccc}
\mathbf{x=0}: & 0+1 & = & 4A & + & -2B & + & C \\
 & & = & 8/9 & - & 2B & + & 5/3
\end{array}$$

so $B = (8/9 + 15/9 - 1)/2 = 7/9$.

Solution: $\dfrac{x^2+1}{(x+1)(x-2)^2} = \dfrac{2}{9(x+1)} + \dfrac{7}{9(x-2)} + \dfrac{5}{3(x-2)}.$

Worked Example 15 *Separate* $\dfrac{x-1}{(x+1)(x^2+1)}$ *into partial fractions.*

Solution The is one linear factor and one quadratic factor, so our solution will be:

$$\frac{x-1}{(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}. \quad (1)$$

Multiply top and bottom of both sides of (1) by the denominator of the right-hand-side to get:

$$x+1 = (x^2+1)A + (x-1)(Bx+C).$$

Evaluate at suitable x values

$$\begin{array}{rcccc}
\mathbf{x=1}: & 2 & = & 2A & + & 0
\end{array}$$

so $A = 1$

There are no more denominator roots, so choose any 'easy' values:

$$\begin{array}{rcccc}
\mathbf{x=0}: & 1 & = & 1\times 1 & + & -1\times C
\end{array}$$

so $C = 0$

$$\begin{array}{rcccc}
\mathbf{x=-1}: & 0 & = & 2\times 1 & + & -2(-B) \\
 & & = & 2 & + & 2B
\end{array}$$

so $B = -1$.

Solution: $\dfrac{x-1}{(x+1)(x^2+1)} = \dfrac{1}{x-1} - \dfrac{x}{x^2+1}.$

2.3.2 Improper Fractions

When the highest power of x in the fraction $P(x)/P'(x)$ does not occur in the denominator we divide $P'(x)$ into $P(x)$ using the method of section 2.2 and then apply the methods described in 2.3.1.

Worked Example 16 *Separate* $\dfrac{3x^3+2x-1}{(x-1)^2}$ *into a polynomial plus a partial fraction expansion.*

Solution There is a repeated linear factor but this gives a highest power in the denominator of x^2 while the numerator has x^3, so we divide:

$$\begin{array}{r|lllllllll}
 & 3x & + & 6 & & & & \\
\hline
x^2-2x+1 & 3x^3 & + & 0x^2 & + & 2x & - & 1 \\
 & 3x^3 & - & 6x^2 & + & 3x & & \\
 & & & 6x^2 & - & x & & \\
 & & & 6x^2 & - & 12x & + & 6 \\
 & & & & & 11x & - & 7
\end{array}$$

The remainder is $11x-7$ so, comparing with (2.4) in section 2.2 we write:

$$\frac{3x^3+2x-1}{(x-1)^2} = 3x+6+\frac{11x-7}{x^2-2x+1} = 3x+6+\frac{11x-7}{(x-1)^2}.$$

We must now separate $\dfrac{11x-7}{(x-1)^2}$ into partial fractions. Since there is a single repeated linear factor we will have:

$$\frac{11x-7}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}.$$

Following the usual 'cross-multiplication' procedure we get:

$$11x-7 \quad = \quad (x-1)A \quad + \quad B.$$

Evaluate at suitable x values

$\boldsymbol{x=1}$: $\quad 4 \quad = \quad 0 \quad + \quad B$

so $B=4$

There are no more denominator roots, so choose any 'easy' value:

$\boldsymbol{x=0}$: $\quad -7 \quad = \quad -1\times A \quad + \quad 4$

so $A=11$

Solution: $\dfrac{3x^3+2x-1}{(x-1)^2} = 3x+6+\dfrac{11}{x-1}+\dfrac{4}{(x-1)^2}.$

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