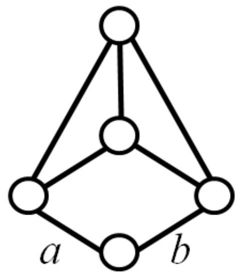
Question set for Chapter 9

1. The aim is to count colourings for the graph G shown below right.

   (a) Explain briefly why the number of λ-colourings of G is zero when $\lambda = 1$ or 2.

   (b) State the number of 3-colourings of G, considering G as a labelled graph, giving a combinatorial explanation for your answer.

   (c) State the number of 4-colourings of G, considering G as a labelled graph, giving a combinatorial explanation for your answer.



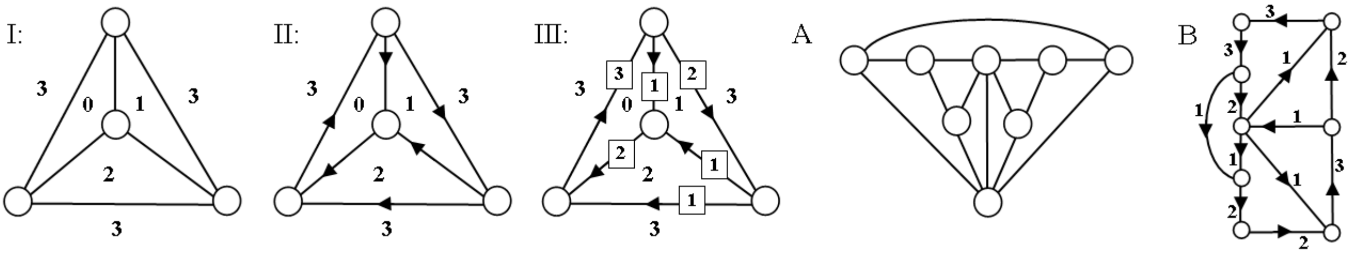
   (d) Use deletion and contraction to compute the chromatic polynomial of G. Your first two choices of edges to use should be a and b, in that order.

   (e) Use the polynomial from part (d) to

      (i) confirm your answers to parts (a)–(c); (ii) calculate $C(G; 5)$ and $C(G; 6)$.

2. Let G be a simple planar graph. Take a plane drawing of G. Suppose it has, say, k faces. Now do the following:

   Step I: Number the faces of G using $0, \ldots, k-1$, for some k, in such a way that no edge belongs to two faces numbered the same. See the example below. Note the 'outside' face is numbered 3: the '3' is repeated several times but it is all one face.



   Step II: Put an arrow on each edge so that if you follow the arrow the larger face number lies on your left.

   Step III: Give each edge a weight which is the difference of its two face numbers. Amazingly, you have what is called a **flow**: at every vertex the sum of the weights flowing **in to** the vertex = the sum of the weights flowing **out of** the vertex!

   (a) Repeat Steps I–III for the graph at A.

   (b) Reverse the Steps I–III for the graph at B: give the outside face the number zero and use the flow to determine the other face numbers, one by one.

   (c) Prove that Steps I–III always give a flow. This is trickier, but an informal proof will do. It is part of some theory which says, among other things, that the Four-Colour Theorem (see Theorem of the Day) is equivalent to saying every planar graph, provided it has no edge whose deletion disconnects the graph, can be given edge directions and a flow using the edge values 1, 2 and 3.