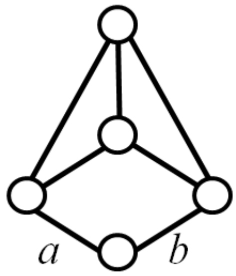
Solutions to Question set for Chapter 9

1. The aim is to count colourings for the graph G shown below right.



   (a) Explain briefly why the number of λ-colourings of G is zero when $\lambda = 1$ or 2.

   (b) State the number of 3-colourings of G, considering G as a labelled graph, giving a combinatorial explanation for your answer.

   (c) State the number of 4-colourings of G, considering G as a labelled graph, giving a combinatorial explanation for your answer.

   (d) Use deletion and contraction to compute the chromatic polynomial of G. Your first two choices of edges to use should be a and b, in that order.

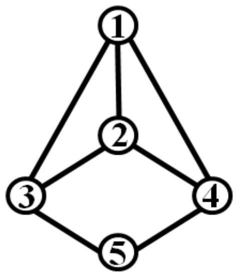
   (e) Use the polynomial from part (d) to

   (i) confirm your answers to parts (a)–(c); (ii) calculate $C(G; 5)$ and $C(G; 6)$.

SOLUTION:

(a) We cannot colour G with 1 colour because there is at least one edge which requires a different colour for each endpoint; we cannot use two colours because there is a triangle in the graph which requires three colours.

(b) The number of 3-colourings is 12. It is convenient to label the vertices of G for reference, as shown on the right, say. Suppose we colour vertex 1 red; then vertex 2 can be blue or green; for either choice, vertex 3 must be the remaining colour and then vertex 4 must be the same as vertex 3. This can be repeated with vertex 1 blue or green instead. So there are 3×2 ways to colour vertices 1 to 4. Finally, for each of these, there are two choices for vertex 5 which can be either colour not used for vertices 3 and 4. So that gives $3 \times 2 \times 2 = 12$ colourings in total.

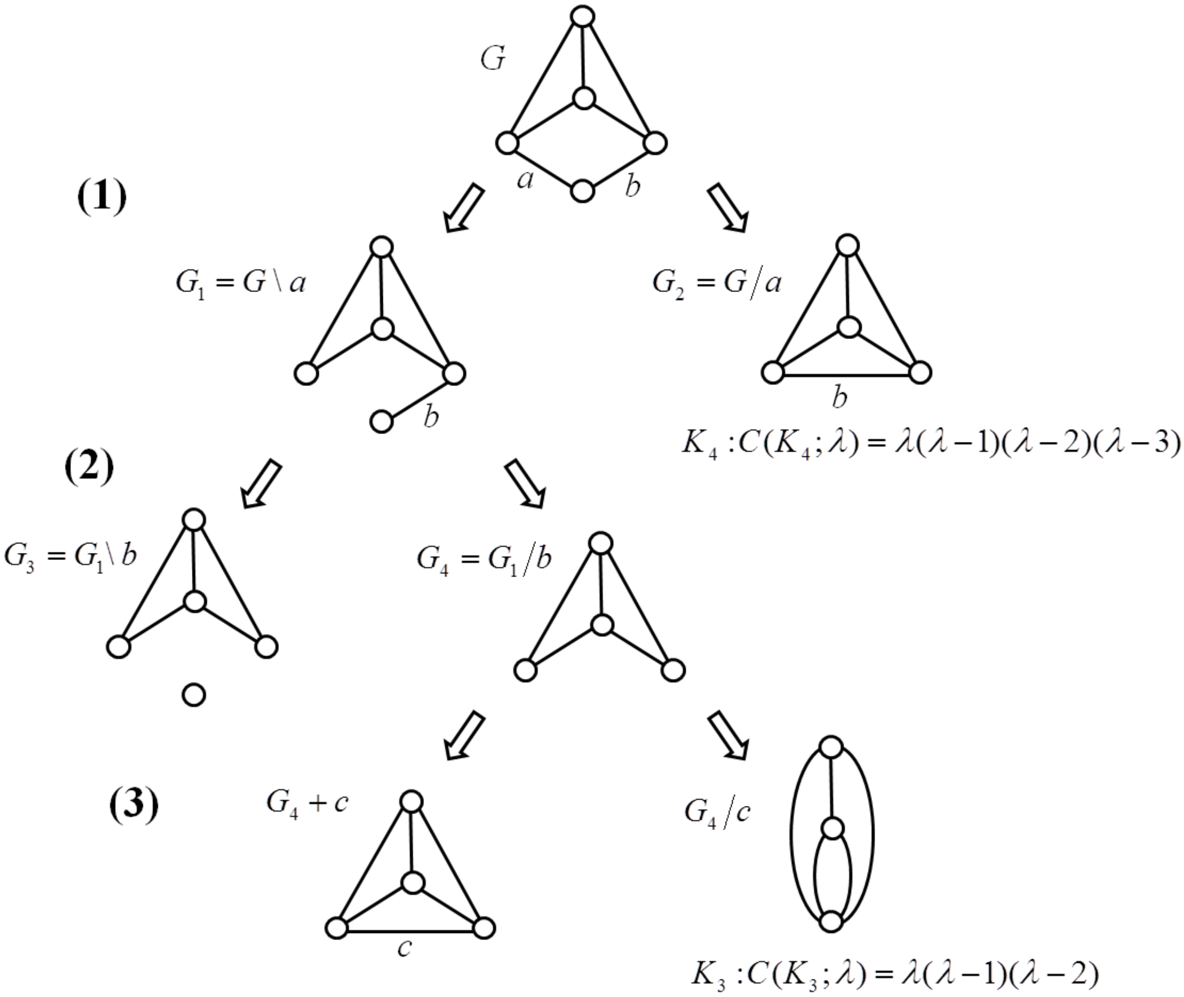


(c) The number of 4-colourings is 120. Using the above labelled version of G again: label 1 has 4 choices, leaving 3 for vertex 2 and then 2 for vertex 3. So there are $4 \times 3 \times 2$ ways to colour vertices 1 to 3.

Now we two choices: (1) vertex 4 is the same colour as vertex 3, in which case 3 colours remain for vertex 5; and (2) vertex 4 is different from vertex 3, in which case only 2 colours remain for vertex 5.

So the grand total of colourings from (1) and (2) is $4\times3\times2\times1\times3+4\times3\times2\times1\times2 = 4 \times 3 \times 2(2 + 3) = 5! = 120$.

(d) The calculation of $C(G;\lambda)$ is best done as shown in the following diagram:



Step (1): we have $C(G;\lambda) = C(G_1;\lambda) - C(G_2;\lambda)$.

Step (2): we have $C(G_1;\lambda) = C(G_3;\lambda) - C(G_4;\lambda)$.

Step (3): we have $C(G_4;\lambda) = C(G_4 + c;\lambda) + C(G_4/c)$ (note the change of sign, using deletion/contraction to *increase* the number of edges).

Let $f = C(G_4;\lambda)$. Then $C(G_3;\lambda) = \lambda C(G_4;\lambda) = \lambda f$ (product of chromatic polynomials of two components).

$G_4 + c = K_4$ and $G_4/c = K_3$ (ignoring multiple edges). So from Step (3):

$$f = \lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2) = \lambda(\lambda-1)(\lambda-2)(1+\lambda-3) = \lambda(\lambda-1)(\lambda-2)^2.$$

Now from Step (2): $C(G_1;\lambda) = \lambda f - f = (\lambda-1)f = \lambda(\lambda-1)^2(\lambda-2)^2$.

Finally, from Step (1) and the fact that $C(G_2;\lambda) = C(K_4;\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)$:

$$\begin{aligned} C(G;\lambda) &= (\lambda-1)f - C(K_4;\lambda) \\ &= \lambda(\lambda-1)^2(\lambda-2)^2 - \lambda(\lambda-1)(\lambda-2)(\lambda-3) \\ &= \lambda(\lambda-1)(\lambda-2)((\lambda-1)(\lambda-2) - (\lambda-3)) \\ &= \lambda(\lambda-1)(\lambda-2)(\lambda^2 - 4\lambda + 5). \end{aligned}$$

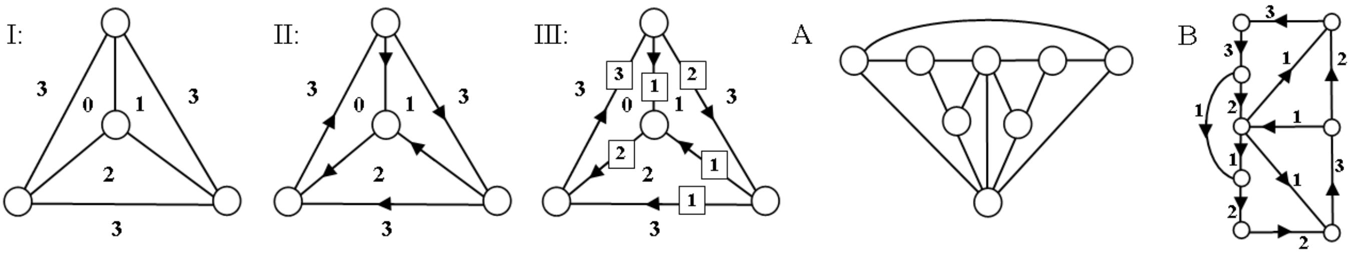
(e) (i) $C(G;\lambda)$ is immediately seen to be zero for $\lambda < 3$.

For $\lambda = 3$ we have $C(G;3) = 3 \times 2 \times 1 \times (9 - 12 + 5) = 12$.

And $C(G;4) = 4\times3\times2\times(16-16+5) = 5!$ (ii) $C(G;5) = 5\times4\times3\times(25-20+5) = 600$.
And $C(G;6) = 6 \times 5 \times 4 \times (36 - 24 + 5) = 2040$.

2. Let G be a simple planar graph. Take a plane drawing of G. Suppose it has, say, k faces. Now do the following:

Step I: Number the faces of G using $0, \ldots, k-1$, for some k, in such a way that no edge belongs to two faces numbered the same. See the example below. Note the 'outside' face is numbered 3: the '3' is repeated several times but it is all one face.



Step II: Put an arrow on each edge so that if you follow the arrow the larger face number lies on your left.

Step III: Give each edge a weight which is the difference of its two face numbers. Amazingly, you have what is called a **flow**: at every vertex the sum of the weights flowing **in to** the vertex = the sum of the weights flowing **out of** the vertex!

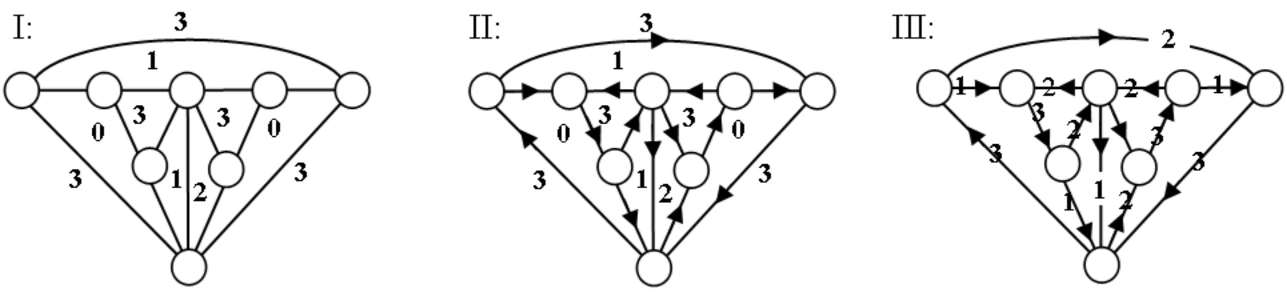
(a) Repeat Steps I–III for the graph at A.

(b) Reverse the Steps I–III for the graph at B: give the outside face the number zero and use the flow to determine the other face numbers, one by one.

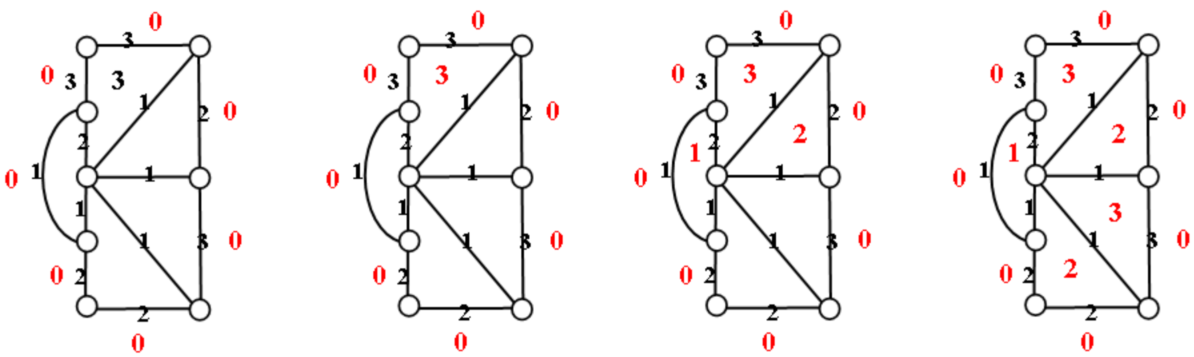
(c) Prove that Steps I–III always give a flow. This is trickier, but an informal proof will do. It is part of some theory which says, among other things, that the Four-Colour Theorem (see Theorem of the Day) is equivalent to saying every planar graph, provided it has no edge whose deletion disconnects the graph, can be given edge directions and a flow using the edge values 1, 2 and 3.

SOLUTION:

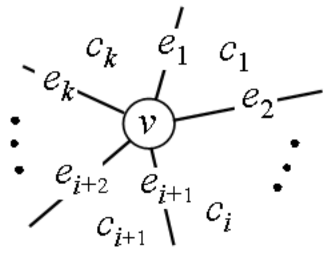
(a) We can find a numbering using $0, \ldots, 3$ as shown in I below. Steps II and III are then fixed:



(b) In the images below we have numbered the outside face 0 and then numbered 1 inside face $3 - 0 = 3$. This numbers two adjacent faces and thereafter the final two faces are fixed.



(c) We must show that the net flow $f^+(v) - f^((v)$ at any vertex v is zero. Suppose $\mathrm{d}(v) = k$. Then k faces of the graph meet at v and these are numbered, say, $c_1, c_2, \ldots, c_k$, as shown right, with the condition that $c_{i-1} \neq c_i \neq c_{i+1}$, with the subscripts taken $\bmod k$ (so c_k is adjacent to c_{k-1} and $c_{k+1} = c_1$).



Now we can calculate

$$f^+(v) - f^-(v) = a_1b_1(c_1 - c_2) + a_2b_2(c_2 - c_3) + \ldots + a_kb_k(c_k - c_1),$$

where $a_1 = \pm 1$ according as $c_i > c_{i+1}$ or not; and $b_i = \pm 1$ according as e_{i+1} is directed out of or into v. But now we observe that $c_i < c_{i+1}$ if and only if e_{i+1} is directed into v, so that $a_ib_i = 1$ for all i. So

$$f^+(v) - f^-(v) = c_1 - c_2 + c_2 - c_3 + \ldots + c_k - c_1 = 0,$$

as required.

(Definitely harder than you will see in the exam, but I hope that the reason it works was discoverable even if the formal explanation might not have been!)