Question set for Chapter 10

1. A set system on ground set $E = \{a, b, c\}$ is given as $\mathscr{A} = \{A_1, A_2, A_3\}$ where

$$A_1 = \{b\},\ A_2 = \{a, c\},\ A_3 = \{b\}.$$

   (a) Use Hall's Theorem to show that no complete transversal for $\mathscr{A}$ exists.

   (b) Suppose that A_1 and A_2 are replaced by $\{a, b, c\}$, while A_3 is replaced by $\{b, c\}$. Explain why $\{a, b, c\}$ is a complete transversal of the resulting set system.

   (c) Use the new version of $\mathscr{A}$ from part (b).

      (i) Draw a bipartite graph, with bipartition $X = E$ and $Y = \mathscr{A}$, representing the incidence relation between E and $\mathscr{A}$.

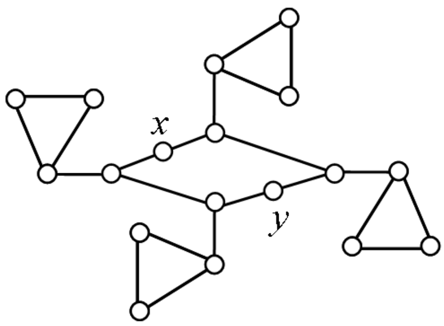
      (ii) Specify a perfect matching for this graph.

      (iii) Write down the 3×3 matrix representing the incidence relation between E and $\mathscr{A}$ and calculate the permanent of this matrix, explaining how this calculation enumerates the perfect matchings of your bipartite graph.

   (d) Give a planar drawing of your bipartite graph from part (c)(i), labelling its vertices $1, \ldots, 6$ in such a way that vertex 1 is adjacent to two vertices: 2 and 6 while vertex 4 is adjacent to two vertices: 3 and 5. Add to your drawing a spanning tree with oriented edges directed away from vertex 1.

   (e) Apply Kastelyn's Theorem using your drawing in part (d) to enumerate perfect matchings in the graph.

2. Suppose G is a undirected graph and U is a subset of the vertices of G. By the *deletion* of U from G we mean removing from G all the vertices in G together with any edges which are joined to vertices in U. Denote by odd(U) the number of connected components which are left which have an odd number of vertices, when you delete U from G. One of the most celebrated theorems in graph theory[1] says that G has a perfect matching if and only if, for every subset U of vertices, odd(U) $\leq |U|$.



   (a) Use the theorem to prove that the graph above-right has no perfect matching

   (b) Give a combinatorial proof that the graph has no perfect matching.

   (c) What if an edge is added joining x to y?

[1]Due to Bill Tutte, 1954