Question set for Chapter 2

1. This question studies derangements using the general-n version of Inclusion-Exclusion:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n}\left((-1)^{k+1}\sum_{\substack{X\subseteq[n]\\|X|=k}}\left|\bigcap_{j\in X} A_j\right|\right). \tag{1}$$

   (a) Write this out in full for $n = 4$: i.e. $|A_1 \cup A_2 \cup A_3 \cup A_4| = \ldots$.

   (b) Use your answer to (a) to find an approximate count of the number of primes less than 100 (see the example in Week 2 lecture notes).

   (c) Now for derangements: suppose A_i denotes #permutations of $[n]$ which do not move i.

      (i) Give an expression in n for $|A_i|$.

      (ii) Given an expression in n for $|A_1 \cap A_2|$.

      (iii) Given an expression in n and $|X|$ for $\left|\bigcap_{i\in X} A_i\right|$, where $X \subseteq [n]$.

   (d) Apply Inclusion-Exclusion (equation 1) to get an expression for the number of permutations of $[n]$ which fix at least one element of $[n]$. (See the count of surjections in Week 2 lecture notes).

   (e) Use (d) to show that the proportion of permutations of $[n]$ that are derangements approaches $e^{-1} = 0.367879\ldots$ as $n \to \infty$. (Hint: use the series expansion $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$).

2. We will use the Pigeon Hole Principle to show that a certain function is not $1-1$ and this will prove:

   Lemma (Sved–Vázsonyi)[1] Suppose we have a sequence of integers $a_1, a_2, \ldots, a_n$, (not necessarily distinct). Then there is some consecutive subsequence $a_k, a_{k+1}, \ldots, a_{k+s}$ with the sum of its entries being divisible by n.

   (a) Check this is true for the sequence $1,\ 3,\ 2,\ 3,\ 3,\ -1,\ 4$ (with $n = 7$).

   (b) For the sequence $a_1, a_2, \ldots, a_n$, define the function $f : [n] \to \{0, 1, \ldots, n-1\}$ by $f(m) = a_1 + \ldots + a_m \bmod n$.

      (i) Show that f is not $1-1$ on the sequence in part (a).

      (ii) Looking back to part (a), what do you notice about the values of m which are mapped by f to the same remainders?

      (iii) Prove that, if f does not map any value to zero then it cannot be $1-1$ (Pigeon Hole Principle!)

   (c) You should now be able to write out a complete proof of the Lemma (tricky but good practice!)

[1] The great Paul Erdős apparently attributed this lemma to Andrew Vázsonyi and Marta Sved. Sved wrote an interesting 'novel' (*Journey into Geometries*, MAA, 1997) in which she explains non-Euclidean geometry in the form of a follow-up to Lewis Carroll's *Alice in Wonderland*; you can read about Vázsonyi here en.wikipedia.org/wiki/Andrew_Vázsonyi.