Solutions for Question set for Chapter 3

1. We will solve the recurrence relation

$$a_n = 3a_{n-2} + 2a_{n-3} - 4n, n \geq 3 \qquad (1)$$
$$\text{initial conditions: } a_0 = 1,\ a_1 = 6,\ a_2 = 12. \qquad (2)$$

(a) Calculate the values of a_3, a_4 and a_5.

(b) Suppose the sequence $b_n = pn + q$ solves the recurrence (1). Find the values of p and q.

(c) Derive the characteristic equation for the homogeneous equation

$$a_n - 3a_{n-2} - 2a_{n-3} = 0. \qquad (3)$$

(d) Find solutions r_1, r_2 and r_3 for the characteristic equation in part (c) to identify a sequence

$$h_n = \alpha r_1^n + \beta n r_2^n + \gamma r_3^n$$

which solves the homogeneous equation (3).

(e) Use the initial conditions (2) to find values of α, β and γ such that $h_n + b_n$ solves (1) and (2). Hence write a final solution $a_n = h_n + b_n$.

(f) Confirm your solution in part (e) against the values you calculated in part (a).

SOLUTION:

(a) We have

$$\begin{aligned} a_3 &= 3a_1 + 2a_0 - 4 \times 3 = 3 \times 6 + 2 \times 1 - 12 = 8; \\ a_4 &= 3a_2 + 2a_1 - 4 \times 4 = 3 \times 12 + 2 \times 6 - 16 = 32; \\ a_5 &= 3a_3 + 2a_2 - 4 \times 5 = 3 \times 8 + 2 \times 12 - 20 = 28. \end{aligned}$$

(b) Substituting for b_n in (1):

$$\begin{aligned} pn + q &= 3(p(n-2) + q) + 2(p(n-3) + q) - 4n \\ \text{so } 0 &= -pn + 3pn + 2pn - 4n \\ & \quad -q - 6p + 3q - 6p + 2q \\ \text{equate coefficients: } 0 &= 4p - 4 \Rightarrow p = 1 \\ \text{and } 0 &= -12p + 4q \Rightarrow 0 = -12 + 4q \Rightarrow q = 3. \end{aligned}$$

So $p = 1,\ q = 3$ and $b_n = n + 3$.

(c) The characteristic equation for the homogeneous equation $a_n - 3a_{n-2} - 2a_{n-3} = 0$ is

$$r^3 - 3r - 2 = 0$$

(Assume solution $a_n = r^n$; then $r^n - 3r^{n-2} - 2r^{n-3} = 0$.
Now divide through by r^{n-2}.)

(d) The characteristic equation factorises as $(r+1)^2(r-2)=0$
(Factor Theorem: $(-1)^3 - 3(-1) - 2 = 0$ and $2^3 - 3(2) - 2 = 0$).
So take $r_1 = r_2 = -1$ and $r_3 = 2$.
And $h_n = \alpha(-1)^n + \beta n(-1)^n + \gamma 2^n$.

(e) Evaluating $h_0 + b_0, h_1 + b_1$ and $h_2 + b_2$ and equating to $a_0 = 1, a_1 = 6$ and $a_2 = 12$:

$$\begin{aligned}
n = 0 : &\quad \alpha(-1)^0 + \beta\times 0\times(-1)^0 + \gamma 2^0 + 3 = 1 \Rightarrow \alpha + \gamma = -2 && (4)\\
n = 1 : &\quad \alpha(-1)^1 + \beta\times 1\times(-1)^1 + \gamma 2^1 + 4 = 6 \Rightarrow -\alpha - \beta + 2\gamma = 2 && (5)\\
n = 2 : &\quad \alpha(-1)^2 + \beta\times 2\times(-1)^2 + \gamma 2^2 + 5 = 12 \Rightarrow \alpha + 2\beta + 4\gamma = 7 && (6)\\
\text{Now} & \\
(4)+(5) : &\quad -\beta + 3\gamma = 0 && (7)\\
(5)+(6) : &\quad \beta + 6\gamma = 9 && (8)\\
(7)+(8) : &\quad 9\gamma = 9. && (9)
\end{aligned}$$

So $\gamma = 1$, giving $\beta = 3$ and finally $\alpha = -3$.
So our solution to (1) is

$$\begin{aligned}
a_n &= -3(-1)^n + 3n(-1)^n + 2^n + n + 3\\
&= 3(n-1)(-1)^n + 2^n + n + 3.
\end{aligned}$$

(f) Calculating for $n = 3, 4, 5$:

$$\begin{aligned}
a_3 &= 3(3-1)(-1)^3 + 2^3 + 3 + 3 = 8;\\
a_4 &= 3(4-1)(-1)^4 + 2^4 + 4 + 3 = 32;\\
a_5 &= 3(5-1)(-1)^5 + 2^5 + 5 + 3 = 28.
\end{aligned}$$

2. Let D_n, $n \geq 1$, denote the number of derangements of $[n] = \{1, 2, \ldots, n\}$. We will derive and solve a recurrence relation for D_n.

   (a) Let δ be a derangement having the property that δ maps some $k \neq 1$ to 1.

      (i) If there are D_s possible derangements δ which also map 1 back to k, then what is the value of s?

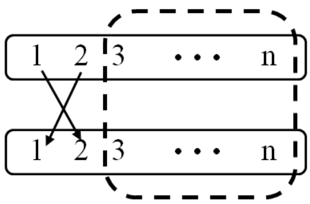
      (ii) Suppose δ does **not** map 1 back to k. If there are D_t possible derangements δ in this case, find the value of t (Hint: suppose 1 is *initially* mapped to k; how many derangements would then rearrange the result to avoid this?)

      (iii) Use parts (i) and (ii) to write D_n as a recurrence relation in terms of D_s and D_t. You will need to factor in the number of possible choices for k.

   (b) Prove that $D_n = n! \displaystyle\sum_{k=0}^{n} \frac{(-1)^k}{k!}$ by showing that this solves the recurrence you found in part (a).

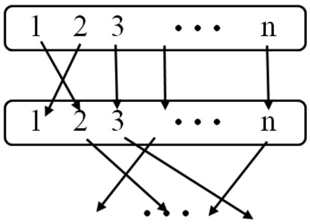
SOLUTION:

(a) Suppose for simplicity that $k = 2$. Then δ in (i) looks like:



(i) There are $n-2$ numbers to rearrange as a derangement of $3, \ldots, n$. There are D_{n-2} ways to do this, so $s = n-2$.

(ii) Suppose we have the same picture as before but we must **avoid** mapping 1 to 2. We can achieve this by first applying the permutation which *only* swaps 1 and 2, leaving everything else fixed, and following this with a new derangement applied to everything except 1:



This can be any derangement applying to $n-1$ objects, so the number of possible δ in this case is D_{n-1}, so $t = n-1$.

(iii) Now D_n moves 1 to some choice of k and then we are either in case (i) or case (ii). So for a given k (by the Summation Principle) we have $D_{n-2} + D_{n-1}$ derangements.
But there are $n-1$ choices for k so (by the Product Principle) there are $(n-1)(D_{n-2} + D_{n-1})$ derangements in total:

$$D_n = (n-1)(D_{n-2} + D_{n-1}), n \geq 2, D_0 = 1.$$

(b) We substitute using $D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$ into the right-hand side of our recurrence and show that it produces the formula for D_n:

$$\begin{aligned}
& (n-1)\left((n-2)! \sum_{k=0}^{n-2} \frac{(-1)^k}{k!} + (n-1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!}\right) \\
= \; & (n-1)\left((n-2)! \sum_{k=0}^{n-2} \frac{(-1)^k}{k!}(1 + (n-1)) + (n-1)! \frac{(-1)^{n-1}}{(n-1)!}\right) \\
= \; & (n-1)! \sum_{k=0}^{n-2} \frac{(-1)^k n}{k!} + (n-1)(-1)^{n-1} \\
= \; & n! \sum_{k=0}^{n-2} \frac{(-1)^k}{k!} + n(-1)^{n-1} - (-1)^{n-1} \\
= \; & n! \sum_{k=0}^{n-2} \frac{(-1)^k}{k!} + \frac{n!}{(n-1)!}(-1)^{n-1} + (-1)^n \\
= \; & n! \sum_{k=0}^{n-2} \frac{(-1)^k}{k!} + \frac{n!}{(n-1)!}(-1)^{n-1} + \frac{n!}{n!}(-1)^n \\
= \; & n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.
\end{aligned}$$