Solutions for question set for Chapter 5

1. A **partition** of a positive integer n is an expression of n as an (unordered) sum of positive integers. For example $1 + 3 + 3$ is a partition of $n = 7$, while $1 + 1 + 1 + 3 + 3 + 3$ is a partition of $n = 12$. The word **unordered** means that, say, $1 + 3 + 3$ and $3 + 1 + 3$ count as the same partition. In 1748 Euler proved a striking identity: the number $d(n)$ of partitions of n into distinct positive integers is the same as the number $o(n)$ of partitions into odd positive integers.

   (a) Confirm that $d(7) = o(7) = 5$ by writing down these partitions.

   (b) Write down the 19 terms in the expansion of

$$(1+a)\left(1+b^2\right)\left(1+c^3\right)\left(1+d^4\right)\left(1+e^5\right)\left(1+f^6\right)\left(1+g^7\right)$$

   which have degree at most 7. (So ab^2c^3, which has degree 6, will be included but not $b^2 f^6$ with degree 8, etc.) Hence write down a brief explanation of why the generating function for $d(n)$ is given by

$$\sum_{n=0}^{\infty} d(n)x^n = \prod_{k=1}^{\infty}\left(1+x^k\right). \tag{1}$$

   (c) Rewrite the right-hand side of (1) as an infinite product of ratios in which each numerator is a difference of two squares. Hence deduce that

$$\prod_{k=1}^{\infty}\left(1+x^k\right) = \frac{1}{(1-x)}\frac{1}{(1-x^3)}\frac{1}{(1-x^5)}\cdots. \tag{2}$$

   (d) Use the Binomial Theorem to rewrite the right-hand side of (2) as a product of infinite series. Then rewrite each power of x in the first series in the form $x^{1+1+\ldots+1}$, each power in the second series in the form $x^{3+3+\ldots+3}$, etc. Hence write down a brief explanation of why the generating function for $o(n)$ is given by

$$\sum_{n=0}^{\infty} o(n)x^n = \prod_{k=1}^{\infty}\frac{1}{(1-x^{2k-1})}, \tag{3}$$

   and why this proves Euler's identity.

SOLUTION:

(a) Distinct parts: $1 + 2 + 4$, $3 + 4$, $2 + 5$, $1 + 6$, 7.
Odd parts: $1 + 1 + 1 + 1 + 1 + 1 + 1$, $1 + 1 + 1 + 1 + 3$, $1 + 3 + 3$, $1 + 1 + 5$, 7.

(b) We have

$$\begin{aligned}
& (1+a)\left(1+b^2\right)\left(1+c^3\right)\left(1+d^4\right)\left(1+e^5\right)\left(1+f^6\right)\left(1+g^7\right)\\
= \; & 1 + a + b^2 + ab^2 + c^3 + ac^3 + d^4 + ad^4 + b^2c^3 + e^5 + ab^2c^3 + b^2d^4 + ae^5 + f^6\\
& + ab^2d^4 + c^3d^4 + b^2e^5 + af^6 + g^7.
\end{aligned}$$

We observe that every term has distinct powers and that any partition of n into distinct parts will match a unique term in the expansion. Since this will still be true if we write $x.x^2$ instead of ab^2, etc, we have the required equality.

(c) Rewriting:

$$\begin{aligned}
\prod_{k=1}^{\infty}\left(1+x^k\right) &= (1+x)\left(1+x^2\right)\left(1+x^3\right)\cdots \\
&= \frac{(1+x)(1-x)}{(1-x)}\frac{(1+x^2)(1-x^2)}{(1-x^2)}\frac{(1+x^3)(1-x^3)}{(1-x^3)}\cdots \\
&= \frac{\cancel{(1-x^2)}}{(1-x)}\frac{\cancel{(1-x^4)}}{\cancel{(1-x^2)}}\frac{\cancel{(1-x^6)}}{(1-x^3)}\frac{\cancel{(1-x^8)}}{\cancel{(1-x^4)}}\cdots \\
&= \frac{1}{(1-x)}\frac{1}{(1-x^3)}\frac{1}{(1-x^5)}\cdots .
\end{aligned}$$

(d) The Binomial Theorem gives $(1-x^n)^{-1} = 1 + x^n + (x^n)^2 + (x^n)^3 + \ldots$. So

$$\begin{aligned}
\frac{1}{(1-x)}\frac{1}{(1-x^3)}\frac{1}{(1-x^5)}\cdots &= (1+x+x^2+x^3+\ldots)(1+x^3+x^6+\ldots)(1+x^5+x^{10}+\ldots)\cdots \\
&= (1+x^1+x^{1+1}+x^{1+1+1}+\ldots)(1+x^3+x^{3+3}+\ldots) \\
&\qquad (1+x^5+x^{5+5}+\ldots)\cdots
\end{aligned}$$

Now every term in the expansion of this product consists of a power of x in the form $x^{1+1+\ldots+1}x^{3+3+\ldots+3}x^{5+5+\ldots+5}\cdots$. And every odd partition of n will correspond to a unique term in the expansion so $\prod_{k=1}^{\infty}\frac{1}{(1-x^{2k-1})}$ is the generating function for odd partitions.

Since it is equal to the generating function for partitions into distinct parts we have $d(n)x^n = o(n)x^n$ for all n and hence $d(n) = o(n)$ for all n.

2. Construct an bijective proof of Euler's identity:

Odd parts: Given a partition into odd parts: $a_1 \times 1 + a_2 \times 3 + a_3 \times 5 + \ldots$, write each a_i as a sum of powers of 2: $(1 + 2 + 2^2 + \ldots)$. Now expand out brackets, keeping terms separate.

(a) Apply this to the odd partition of 100 given by

$$1+1+1+3+3+3+5+5+5+5+5+5+7+9+9+11+11+11$$

to get a partition of 100 into distinct parts.

(b) Explain briefly *why* the parts in part (a) are distinct.

Distinct parts: Try to deduce this for yourself:

(c) Write down how to reverse the injection odd→distinct just specified. Show that this works for your answer to part (a) above.

SOLUTION:

(a) We have

$$\begin{aligned}
& 1+1+1+3+3+3+5+5+5+5+5+5+7+9+9+11+11+11 \\
= \;& 3\times 1 + 3\times 3 + 6\times 5 + 1\times 7 + 2\times 9 + 3\times 11 \\
= \;& (1+2)\times 1 + (1+2)\times 3 + (2+4)\times 5 + 1\times 7 + 2\times 9 + (1+2)\times 11 \\
= \;& 1+2+3+6+10+20+7+18+11+22.
\end{aligned}$$

(b) The terms must be distinct because we cannot have $2^k p = 2^{k'} q$ for distinct odd p and q (otherwise, suppose $k < k'$; dividing by 2^k would leave $p = 2^{k'-k} q$ equating an odd and even number).

(c) Write each even term in the partition uniquely in the form $2^k p$ for an odd number p. Now rewrite $2^k p$ as $\overbrace{p + p + \ldots + p}^{2^k \text{ times}}$.

For $1 + 2 + 3 + 6 + 10 + 20 + 7 + 18 + 11 + 22$ we get

$$\begin{aligned}
&1 + 2 \times 1 + 3 + 3 \times 2 + 2 \times 5 + 4 \times 5 + 7 + 2 \times 9 + 11 + 2 \times 11 \\
&\quad = 1 + 1 + 1 + 3 + 3 + 3 + 5 + 5 + 5 + 5 + 5 + 5 + 7 + 9 + 9 + 11 + 11 + 11.
\end{aligned}$$