Solutions for question set for Chapter 6

1. We will use the method of generating functions to solve a degree 3 linear homogeneous recurrence relation (which we could equally solve by the Chapter 3 approach, but now we get something extra: a closed form for the generating function of the recurrence).

   Let $(a_n)_{n\geq 0}$ be defined by

   $$a_n = 8a_{n-1} - 21a_{n-2} + 18a_{n-3},\ n \geq 3, \qquad a_0 = -1,\ a_1 = 1,\ a_2 = 4. \tag{1}$$

   (a) Use equation (1) to write down the generating function $A(x)$ for the sequence $(a_n)_{n\geq 0}$, up to the term in x^5.

   (b) Suppose $C(x)$ is the generating function defined by

   $$c_n = a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} \tag{2}$$

   and that $C(x)$ can be expressed as $A(x)D(x)$ for some unknown generating function $D(x) = \sum_{k=0}^{\infty} d_k x^k$.

   (i) Write down an expression for c_n as a sum of products of the a_ks and d_ks (see Week 6, Lecture 1 notes).

   (ii) By equating terms with (2), find the values of d_k, $k = 0, 1, 2, 3$. Hence, by applying your expression from part (i) for $n = 0, 1, 2$, find the values of c_0, c_1 and c_2.

   (iii) Explain briefly why $c_n = 0$ for $n > 2$, and $d_n = 0$ for $n > 3$.

   (c) Use your answers from part (b) to express $A(x)$ in closed form, as a ratio of two polynomials: a **rational function**.

   (d) Assuming your answer to part (c) is correct, the rational function you found may be expressed, via factorisation and partial fractions separation, as

   $$\frac{-11}{1-2x} - \frac{7}{3(1-3x)^2} + \frac{37}{3(1-3x)}.$$

   (You can check if you have time, by putting the above sum over a common denominator which you then expand out).

   Use the binomial theorem to produce a closed formula for a_n, $n \geq 0$, and check that this gives the same generating function $A(x)$ as you wrote down in part (a).

SOLUTION:

(a) You should get

$$\begin{aligned} a_3 &= 8\times 4 - 21\times 1 + 18\times -1 = -7;\\ a_4 &= 8\times -7 - 21\times 4 + 18\times 1 = -122;\\ a_5 &= 8\times -122 - 21\times -7 + 18\times 4 = -757. \end{aligned}$$

So we have

$$A(x) = -1 + x + 4x^2 - 7x^3 - 122x^4 - 757x^5 - \dots.$$

(b) (i) This comes directly from page 1 of the notes:

$$c_n = a_0 d_n + a_1 d_{n-1} + \ldots + a_{n-2} d_2 + a_{n-1} d_1 + a_n d_0.$$

(ii) Again, the procedure is just the same as in Chapter 6, section 1, using equation (2):

$$\begin{aligned}
a_n &= a_n d_0, \quad \text{so} \quad d_0 = 1, &\Rightarrow\quad & c_0 = a_0 d_0 = -1 \times 1 = -1; \\
-8a_{n-1} &= a_{n-1} d_1, \quad \text{so} \quad d_1 = -8, &\Rightarrow\quad & c_1 = a_0 d_1 + a_1 d_0 = -1 \times -8 + 1 \times 1 = 9; \\
21a_{n-2} &= a_{n-2} d_2, \quad \text{so} \quad d_2 = 21, &\Rightarrow\quad & c_2 = a_0 d_2 + a_1 d_1 + a_2 d_0 \\
&&& \quad = -1 \times 21 + 1 \times -8 + 4 \times 1 = -25; \\
-18a_{n-3} &= a_{n-3} d_3, \quad \text{so} \quad d_3 = -18, &\Rightarrow\quad & c_3 = a_0 d_3 + a_1 d_2 + a_2 d_1 + a_3 d_0 \\
&&& \quad = -1 \times -18 + 1 \times 21 + 4 \times -8 + -7 \times 1 = 0. \\
&&& \text{(but we know } c_3 = 0 \text{ — see below).}
\end{aligned}$$

(iii) $c_n = 0$ for $n > 2$ because $c_n = a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3}$ by (2) and this is zero for $n \geq 3$ by (1).
And if we continue equating coefficients as in part (ii) we get $0 \times a_{n-3} = a_{n-3} d_3$, so $d_4 = 0$, etc.

(c) Since $A(x)D(x) = C(x) = -1 + 9x - 25x^2$, and $D(x) = 1 - 8x + 21x^2 - 18x^3$ (the characteristic equation), we have

$$A(x) = \frac{-1 + 9x - 25x^2}{1 - 8x + 21x^2 - 18x^3}.$$

(d) Now we are told that we can rewrite the answer to (c) as

$$A(x) = \frac{-11}{1 - 2x} - \frac{7}{3(1 - 3x)^2} + \frac{37}{3(1 - 3x)}.$$

Applying the binomial theorem this gives:

$$\begin{aligned}
A(x) &= -11 \sum_{k=0}^{\infty} 2^k x^k - \frac{7}{3} \sum_{k=0}^{\infty} (k+1) 3^k x^k + \frac{37}{3} \sum_{k=0}^{\infty} 3^k x^k \\
&= -11 \sum_{k=0}^{\infty} 2^k x^k - \frac{7}{3} \sum_{k=0}^{\infty} k 3^k x^k + \left(-\frac{7}{3} + \frac{37}{3}\right) \sum_{k=0}^{\infty} 3^k x^k \\
&= -11 \sum_{k=0}^{\infty} 2^k x^k - 7 \sum_{k=0}^{\infty} k 3^k x^{k-1} + 10 \sum_{k=0}^{\infty} 3^k x^k
\end{aligned}$$

Which gives a closed formula for the n-th term: $a_n = -11\,(2^n) + (30 - 7n)3^{n-1}$.
We check each a_n, $n = 0, \ldots, 5$, e.g. $n = 0 : a_0 = -11\left(2^0\right) + (30 - 0)3^{-1} = -11 + 10 = -1$, etc.

2. There are two related parts, using exponential generating functions to count quite different combinatorial objects:

(a) Firstly, we get an exponential generating function (egf) for the number of 2-partions of $[n]$, i.e. for the numbers $S(n, 2)$, where $S(n, k)$ are the Stirling numbers of the 2nd kind (see Week 6, Lecture 3).

(i) Write down the closed form egf for nonempty subsets of $[n]$.

(ii) Use your answer to (i) to get the egf for 2-partitions, dividing $[n]$ into *two non-empty* subsets. Make sure that your modification also adjusts for the fact that 2-partitions are not *ordered*, so that $\{1, 3\}, \{2, 4, 5\}$ and $\{2, 4, 5\}, \{1, 3\}$ are the *same* 2-partition of $[5]$.

(iii) By expanding out brackets in your answer to part (ii), and then by writing out the appropriate power series expansions, give a closed formula for $S(n, 2)$.

(b) Now we get the egf for permutations which are **involutions**, i.e. whose squares are the identity permutation. An example on [5] would be 2, 1, 3, 4, 5, since swapping 1 and 2, and then swapping them again, takes us back to the start: 1, 2, 3, 4, 5.

Involutions are easier to recognise in **disjoint cycle notation** in which the above permutation would be written (1 2)(3)(4)(5) (or just (1 2) if we ignore cycles of length 1). For instance, there are four involutions on [3]:

$$(1)(2)(3),\ (1\ 2)(3),\ (1\ 3)(2),\ \text{ and } (1)(2\ 3).$$

(i) Write down the **ten** involutions on [4] in disjoint cycle notation.

(ii) What is the egf for (trivially) arranging a set as a 1-element set or a 2-element set. (Hint: look at equation (3) in the Week 6, Lecture 3 notes).

(iii) Fix N. What is the egf for partitioning $[n]$ into N subsets of sizes 1 or 2. Make sure you adjust for the fact that the partition is *unordered* (see part (a)(ii)).

(iv) Summing over all N, you should be able to write your answer to part (iii) as a product of two exponential series. By expanding out enough terms of this product, confirm that your egf correctly counts involutions on [4].

SOLUTION:

(a) (i) This is given in the Week 6, Lecture 3 lecture notes: $e^x - 1$.

(ii) The Product Principle for egf's gives the egf for a partition into a 'left' nonempty subset and a 'right' nonempty subset as: $(e^x - 1)\times(e^x - 1)$. But this will count every 2-partition twice because each 'left' nonempty subset will also appear later as a 'right' nonempty subset and vice versa. So we must divide by two to get our answer:

$$\text{egf for \#2-partitions } = \frac{1}{2}\left(e^x - 1\right)^2.$$

(iii) Expanding out:

$$\begin{aligned}
\frac{1}{2}\left(e^x - 1\right)^2 &= \frac{1}{2}\left(e^{2x} - 2e^x + 1\right) \\
&= \frac{1}{2}\left(1 + 2x + 2^2\frac{x^2}{2!} + 2^3\frac{x^3}{3!} + \ldots - 2\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) + 1\right) \\
&= \left(2^0 - 1\right)\frac{x^1}{1!} + \left(2^1 - 1\right)\frac{x^2}{2!} + \left(2^2 - 1\right)\frac{x^3}{3!} + \ldots = \sum_{k=1}^{\infty}\left(2^{k-1} - 1\right)\frac{x^k}{k!}.
\end{aligned}$$

So the n-th term gives us $S(n, 2) = 2^{n-1} - 1$, for $n \geq 1$.

(b) (i) The ten involutions on [4] are:

all cycles length 1:	1					
1 cycle length 2:	(1 2)	(1 3)	(1 4)	(2 3)	(2 4)	(3 4)
2 cycles length 2:	(1 2)(3 4)	(1 3)(2 4)	(1 4)(2 3)			

(ii) Trivially arranging $[n]$ as a 2-element set (given as equation (3) in the Week 6, Lecture 3 notes) is $x^2/2!$ because only when $n = 2$ is this possible. If we are allowed a 1-element set as well then we get $x + \dfrac{x^2}{2!}$.

(iii) For fixed N, the egf for partitioning $[n]$ into N subsets of sizes 1 or 2 will be $\dfrac{1}{N!}\left(x + \dfrac{x^2}{2!}\right)^N$, where we divide by $N!$ because there are $N!$ orders in which our 1 and 2 cycles may reappear: (1)(2 3)(4), (2 3)(1)(4), (4)(1)(2 3), etc.

(iv) Summing over N, the egf for partitioning a set into subsets of size 1 or 2 is

$$\sum_{N=0}^{\infty} \frac{1}{N!}\left(x+\frac{x^2}{2!}\right)^N = \sum_{k=0}^{\infty} \frac{(x+x^2/2)^k}{k!} = e^{x+x^2/2}.$$

Writing $e^{x+x^2/2} = e^x \times e^{x^2/2}$ we get a product of two expansions:

$$\begin{aligned} e^x e^{x^2/2} &= \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots\right)\left(1+\frac{x^2}{2}+\frac{x^4}{2^2.2!}+\frac{x^6}{2^3.3!}+\dots\right) \\ &= 1+x+\left(\frac{1}{2}+\frac{1}{2!}\right)x^2+\left(\frac{1}{2}+\frac{1}{3!}\right)x^3+\left(\frac{1}{2^2.2!}+\frac{1}{2!.2}+\frac{1}{4!}\right)x^4+\dots \\ &= 1+x+2\frac{x^2}{2!}+4\frac{x^3}{3!}+10\frac{x^4}{4!}+\dots, \end{aligned}$$

as required.

If you look hard enough at this product of two power series it is possible to spot a formula, although not in closed form, for the general term:

$$a_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{k!(n-2k!)2^k},$$

where $\lfloor n/2 \rfloor$ *is the largest integer not exceeding* $n/2$. *For example, for* $n = 3$, *we have* $\lfloor 3/2 \rfloor = 1$ *and the sum becomes* $\left(\frac{3!}{0!.3!.2^0}+\frac{3!}{1!.1!.2^1}\right) = 1+3 = 4.$

3. Generating functions give a nice way to prove a truly iconic result, due to Leonhard Euler: $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\dots = \frac{\pi^2}{6}$. In other words, if $L(x) = \sum_{k=1}^{\infty}\frac{x^k}{k^2}$, then we want to prove $L(1) = \frac{\pi^2}{6}$.

   (a) Use the standard power series expansion $\ln(1-t) = -t-\frac{t^2}{2}-\frac{t^3}{3}-\dots$ to show that $L(z) = \int_0^z \frac{-\ln(1-t)}{t}\,dt$. (We have switched from x to z because there are complex numbers lurking in the background here...)

   (b) The integrand in (a) does not have an antiderivative (using elementary functions). So the integral is labelled a 'special function' and given a name: the **dilogarithm**, denoted by $\mathrm{Li}_2(z)$. So now we know that $L(z) = \mathrm{Li}_2(z)$ and, in particular, $L(1) = \mathrm{Li}_2(1)$. Euler had an identity involving these too:

$$\mathrm{Li}_2\left(\frac{-1}{z}\right)+\mathrm{Li}_2(-z)+\mathrm{Li}_2(1) = -\frac{1}{2}(\ln z)^2.$$

   Use this to express $\mathrm{Li}_2(1)$ in terms of $\ln(-1)$.

   (c) Use another Euler classic, $e^{i\pi}+1 = 0$, to turn your answer to (b) into the identity $\mathrm{Li}_2(1) = \frac{\pi^2}{6}$.

SOLUTION:

(a) We have:

$$\begin{aligned}
-\ln(1-t) &= t + \frac{t^2}{2} + \frac{t^3}{3} + \dots \\
\text{so} \quad \frac{-\ln(1-t)}{t} &= 1 + \frac{t}{2} + \frac{t^2}{3} + \dots \\
\text{and} \quad \int_0^z \frac{-\ln(1-t)}{t}\,dt &= \int_0^z \left(1 + \frac{t}{2} + \frac{t^2}{3} + \dots\right) dt \\
&= \left[t + \frac{t^2}{2^2} + \frac{t^3}{3^2} + \dots\right]_0^z = z + \frac{z^2}{4} + \frac{z^3}{9} + \dots \\
&= L(z).
\end{aligned}$$

(b) Substituting $z = -1$ in

$$\mathrm{Li}_2\left(\frac{-1}{z}\right) + \mathrm{Li}_2(-z) + \mathrm{Li}_2(1) = -\frac{1}{2}(\ln z)^2$$

gives $\mathrm{Li}_2(1) + \mathrm{Li}_2(1) + \mathrm{Li}_2(1) = -\frac{1}{2}(\ln -1)^2$. So $\mathrm{Li}_2(1) = -\frac{1}{6}(\ln(-1))^2$.

(c) From $e^{i\pi} + 1 = 0$ we can substitute $e^{i\pi}$ for -1 in the answer from (b) to get:

$$\mathrm{Li}_2(1) = -\frac{1}{6}\left(\ln\left(e^{i\pi}\right)\right)^2 = -\frac{1}{6}(i\pi)^2 = -\frac{1}{6} \times -1 \times \pi^2 = \frac{\pi^2}{6}.$$

You can read more about this particular proof of Euler's result at Theorem of the Day.