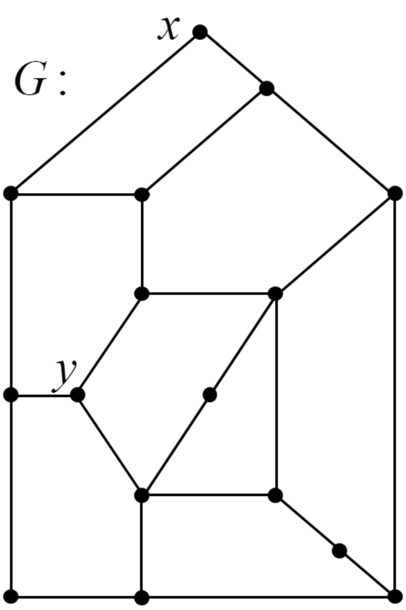
Question set for Chapter 8

1. A graph is called **Hamiltonian** if it has a spanning cycle, that is, a cycle that passes through every vertex exactly once; such a cycle is called a **Hamilton cycle**. Consider the graph G below, right:

   (a) Show by highlighting a suitable set of edges on a copy of the image on the right, that G is Hamiltonian.

   (b) The faces of a planar Hamiltonian graph may be coloured with four colours so that no adjacent faces have the same colour. This is because the faces **interior** to a Hamilton cycle may be alternately coloured, say, red and blue, while the faces **exterior** may be coloured alternately, say, green and yellow. Using your answer to part (a) give a four-colouring of the faces of graph G.

   (c) Although G is planar, the addition of a single edge may be enough to destroy this property. Use Kuratowski's Theorem to show that, if an edge is adding joining the vertices x and y, the resulting graph is not planar. Suggestion: mark two sets, A and B, of three vertices on a copy of the image on the right and highlight edges forming disjoint paths joining all vertices of A to all vertices of B. Explain why this demonstrates non-planarity.



2. A neat theorem due to Emanuel Grinberg gives a necessary condition for a planar graph G to be Hamiltonian. Assume G is Hamiltonian with a Hamilton cycle C (as you found in part (a) of question 1). Take a drawing of G in the plane and for $k \geq 3$ let f_k^- denote the number of faces of G in this drawing which are k-gons (having k edges around the face) and are **interior** to cycle C. Similarly, let f_k^+ denote the number of faces of G which are k-gons **exterior** to cycle C. Then the following identity is satisfied:

$$\sum_{k\geq 3}(k-2)(f_k^- - f_k^+) = 0.$$

   (a) Check that Grinberg's theorem holds for the Hamilton cycle you found for graph G in question 1, part (a).

   (b) Suppose that vertex x is deleted from this graph. Use Grinberg's theorem to show that the resulting graph is no longer Hamiltonian. *[HINT: Grinberg's theorem remains true if the terms of its summation are reduced modulo 3.]*

   (c) We will now use Euler's theorem on planar graphs to prove Grinberg's theorem. Let n denote the number of vertices of G. For a Hamilton cycle C in a graph G, denote by e_C^- the number of edges of G in the interior to C. Denote by G_C^- the subgraph (drawn in the plane) consisting of these e_C^- edges together with the edges of C. Denote by f_C^- the number of faces of G_C^-. Then Euler's theorem says that $n - e_C^- + f_c^- = 2$.

      i. Write f_c^- in terms of the f_k^-. Now (a bit harder) write e_C^- in terms of the f_k^-.

      ii. Rearrange to show that $\sum_{k\geq 3}(k-2)f_k^- = n-2$.

      iii. Repeat these two steps but with e_c^- and f_C^- replaced by e_c^+ and f_C^+, suitably defined.

      iv. Conclude that $(k-2)f_k^- = (k-2)f_k^+$, which is Grinberg's theorem.