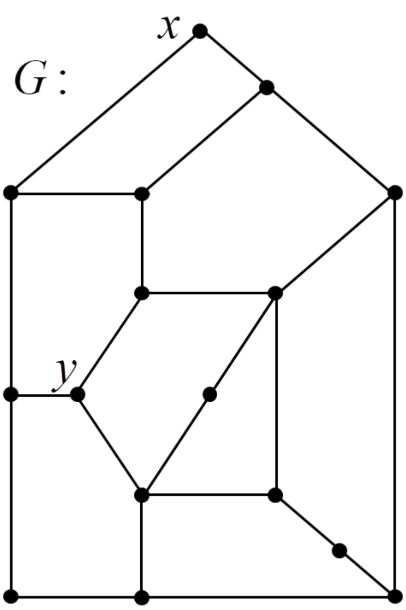
Solutions to Question set for Chapter 8

1. A graph is called **Hamiltonian** if it has a spanning cycle, that is, a cycle that passes through every vertex exactly once; such a cycle is called a **Hamilton cycle**. Consider the graph G below, right:

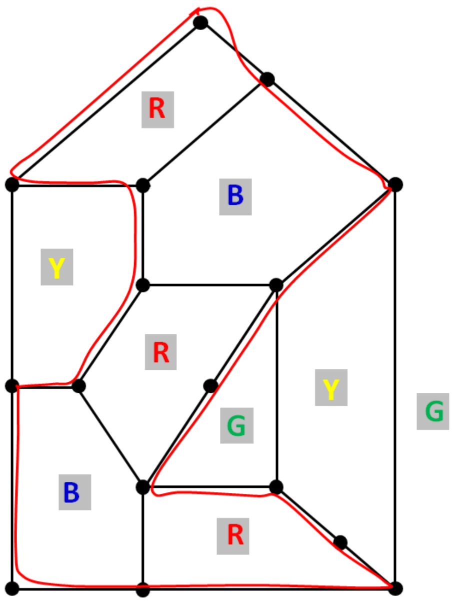
   (a) Show by highlighting a suitable set of edges on a copy of the image on the right, that G is Hamiltonian.

   (b) The faces of a planar Hamiltonian graph may be coloured with four colours so that no adjacent faces have the same colour. This is because the faces **interior** to a Hamilton cycle may be alternately coloured, say, red and blue, while the faces **exterior** may be coloured alternately, say, green and yellow. Using your answer to part (a) give a four-colouring of the faces of graph G.

   (c) Although G is planar, the addition of a single edge may be enough to destroy this property. Use Kuratowski's Theorem to show that, if an edge is adding joining the vertices x and y, the resulting graph is not planar. Suggestion: mark two sets, A and B, of three vertices on a copy of the image on the right and highlight edges forming disjoint paths joining all vertices of A to all vertices of B. Explain why this demonstrates non-planarity.



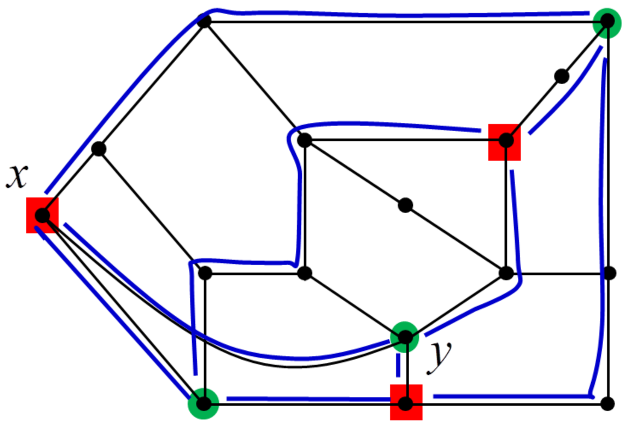
SOLUTION



   (a) The image on the right shows a Hamilton cycle, highlighted as red edges. There are other possible answers.

   (b) The image also shows a four-colouring derived from the Hamilton cycle.

   (c) The image below shows a subdivision of $K_{3,3}$ embedded in the graph G with the additional edge (rotated so it fits on the page). This means, by Kuratowski's Theorem, that the graph cannot be planar.



2. A neat theorem due to Emanuel Grinberg gives a necessary condition for a planar graph G to be Hamiltonian. Assume G is Hamiltonian with a Hamilton cycle C (as you found in part (a) of question 1). Take a drawing of G in the plane and for $k \geq 3$ let f_k^- denote the number of faces of G in this drawing which are k-gons (having k edges around the face) and are **interior** to cycle C. Similarly, let f_k^+ denote the number of faces of G which are k-gons **exterior** to cycle C. Then the following identity is satisfied:

$$\sum_{k\geq 3}(k-2)(f_k^- - f_k^+) = 0.$$

(a) Check that Grinberg's theorem holds for the Hamilton cycle you found for graph G in question 1, part (a).

(b) Suppose that vertex x is deleted from this graph. Use Grinberg's theorem to show that the resulting graph is no longer Hamiltonian. *[HINT: Grinberg's theorem remains true if the terms of its summation are reduced modulo 3.]*

(c) We will now use Euler's theorem on planar graphs to prove Grinberg's theorem. Let n denote the number of vertices of G. For a Hamilton cycle C in a graph G, denote by e_C^- the number of edges of G in the interior to C. Denote by G_C^- the subgraph (drawn in the plane) consisting of these e_C^- edges together with the edges of C. Denote by f_C^- the number of faces of G_C^-. Then Euler's theorem says that $n - e_C^- + f_c^- = 2$.

   i. Write f_c^- in terms of the f_k^-. Now (a bit harder) write e_C^- in terms of the f_k^-.

   ii. Rearrange to show that $\sum_{k\geq 3}(k-2)f_k^- = n-2$.

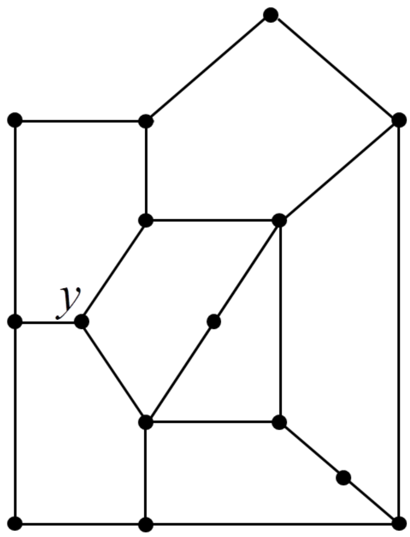
   iii. Repeat these two steps but with e_c^- and f_C^- replaced by e_c^+ and f_C^+, suitably defined.

   iv. Conclude that $(k-2)f_k^- = (k-2)f_k^+$, which is Grinberg's theorem.

SOLUTION:

(a) Nonzero interior k-gons for the indicated Hamilton cycle C are $f_4^- = 1$, $f_5^- = 4$. Nonzero exterior k-gons are $f_4^+ = 1$, $f_5^+ = 2$, $f_8^+ = 1$. So the LHS of Grinberg's theorem is $(4-2)(1-1)+(5-2)(4-2)+(8-2)(0-1) = 0$, as required.

(b) If vertex x is deleted then the graph becomes as shown on the right. We have 4-gons, 5-gons and 8-gons, as before, but no Hamilton cycle is now indicated. If a Hamilton cycle exists then the LHS of Grinberg's theorem will be $2(f_4^- - f_4^+)+3(f_5^- - f_5^+)+6(f_8^- - f_8^+)$. For this to be zero it must also be zero when reduced modulo 3. So a Hamilton cycle exists only if $2(f_4^- - f_4^+) = 0$. But now there is only one 4-gon so this is impossible.



(c) i. We can write $f_c^- = 1 + \sum_{k\geq 3} f_k^-$, since the faces of G_C^- are exactly the interior faces of C plus the single face which is the plane exterior to the graph. Each k-gon interior to C contibutes k edges to e_C^-. But each of these edges will be counted twice since it belongs either to a neighbouring face or to the cycle C. The cycle C contributes n to this double counting because the cycle is Hamilton and includes every vertex of G. So we can write $e_c^- = \left(n + \sum_{k\geq 3} k f_k^-\right)/2$.

ii. Substituting into Euler's theorem:

$$\begin{aligned} n - e_C^- + f_c^- &= 2 \text{ becomes} \\ n - \left(n + \sum_{k\geq 3} k f_k^-\right)/2 + 1 + \sum_{k\geq 3} f_k^- &= 2 \\ n - \sum (k-2) f_k^- + 2 &= 4 \\ \sum (k-2) f_k^- &= n - 2. \end{aligned}$$

iii. The above calculation is the same if we define G_C^+ to be the part of G exterior to cycle C, with e_c^+ being the number of edges lying on faces exterior to C and f_C^+ being the number of these faces.

iv. Then we have $(k-2) f_k^- = n - 2 = (k-2) f_k^+$, as required.