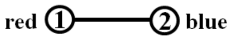
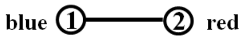
# Chapter 9. Section 1

## Counting Colourings

Let $C(G; \lambda)$ denote the number of ways to colour graph $G$ using $\lambda$ colours so that adjacent vertices are differently coloured. The vertices of $G$ are assumed to be **labelled**: 2-colouring a labelled $K_2$, say, can be done in two ways: 



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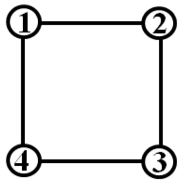



**Example** How many ways are there to colour the labelled copy of the 4-cycle $C_4$ on the right, using

(a) $\lambda = 2$ colours? (b) $\lambda = 3$ colours?

**Solution:** We shall answer with combinatorial arguments. Later we shall find that a purely algebraic solution can also be given.

(a) Suppose ① is coloured red. Then ② and ④ must be blue; then ③ must be red. So there is only one colouring if we start with ①=red. But there is a second colouring starting with ①=blue, so there are **two** 2-colourings in total.

(b) With $\lambda = 3$ we can colour ① with 3 colours, leaving 2 colours for ②. By the Product Principle, #ways to colour ① and ② = $3 \times 2 = 6$.

For the two remaining vertices we consider two exclusive cases and apply the Summation Principle:

**④ =②:** we only allow 1 colour for ④; we find there are now 2 colour choices for ③.

**④ ≠②:** since ④ is also adjacent to ① it again has just 1 colour choice; now ③ is adjacent to two different colours so there is only 1 choice for ③.

So the grand total for number of colours is $6 \times 1 \times 2 + 6 \times 1 \times 1 = 18$.

## Special Cases

1. $C(K_n; \lambda) = \lambda(\lambda - 1)\cdots(\lambda - n + 1)$. This follows by the Product Principle: there are $\lambda$ colours for the first vertex; each choice leaves $\lambda - 1$ for vertex 2; leaving $\lambda - 2$ for vertex 3, etc.
2. $C(N_n; \lambda) = \lambda^n$, where $N_n$ is the **null graph** having $n$ vertices and zero edges. This follows because we are mapping $n$ vertices independently to $\lambda$ colours, so #colourings = #functions $[n] \longrightarrow [\lambda]$ (see Week 3, Lecture 1).
3. $C(T; \lambda) = \lambda(\lambda-1)^{n-1}$, for a tree $T$. This follows by induction on $n$: for $n = 1$ the tree must be equal to $N_1$, so $C(T; \lambda) = C(N_1; \lambda) = \lambda^1 = \lambda(\lambda-1)^{1-1}$, as required. Assume the result is true for $n = K$ and let $T$ have $K + 1$ vertices. Remove one of degree 1 to obtain a tree on $K$ vertices having $\lambda(\lambda - 1)^{K-1}$ colourings by inductive hypothesis. Now the removed vertex may be replaced using any colour except that used for its adjacent vertex. So by the Product Principle, total colours for $T$ is $\lambda(\lambda-1)^{K-1} \times (\lambda-1) = \lambda(\lambda-1)^K$, as required. (Compare with the proof (Chapter 7, Section 2, page 2) that a tree on $n$ vertices has $n-1$ edges).

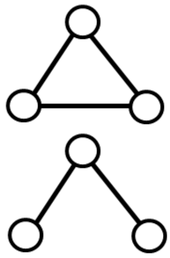
Using the above special cases we can calculate $C(G; \lambda)$ for any graph $G$ using two **recurrence** relations.

**Theorem 1** Suppose $G$ consists of two connected components $G_1$ and $G_2$ (i.e. the $G_i$ are connected graphs but there are no edges between $G_1$ and $G_2$). Then

$$C(G; \lambda) = C(G_1; \lambda) \times C(G_2; \lambda). \qquad (1)$$

**Proof.** This follows directly from the Product Principle since colouring the vertices of $G_1$ is independent of the way $G_2$ is coloured. (Compare with Theorem B for rook polynomials, on p.2 of Chapter 5, Section 1).

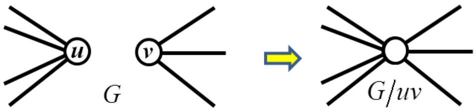
**Example** How many ways are there to $\lambda$-colour the graph $G$ on the right, regarded as a labelled graph?



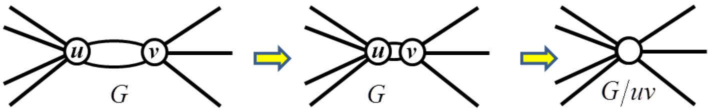
**Solution:** The graph consists of two components, a $K_3$ and a $P_3$. Now $C(K_3, \lambda) = \lambda(\lambda-1)(\lambda-2)$ and $C(P_3, \lambda) = \lambda(\lambda-1)^2$ (since $P_3$ is a tree). So by Theorem 1,

$$C(G, \lambda) = \lambda(\lambda-1)(\lambda-2) \times \lambda(\lambda-1)^2 = \lambda^2(\lambda-1)^3(\lambda-2).$$

The second recurrence relation uses a new graph operation. Given two vertices $u$ and $v$ in a graph $G$, the **contraction** of $u$ and $v$, denoted by $G/uv$ consists of identifying the two vertices, giving a single vertex whose incident edges are the edges of $G$ incident with $u$ or $v$ both not both. Any edges incident with both $u$ and $v$ are 'shrunk' down and disappear. This is illustrated below.



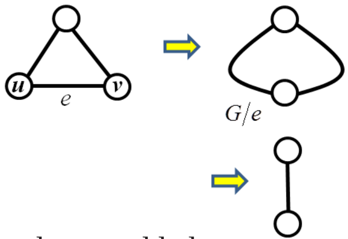

$u$ and $v$ not adjacent: the vertices are just 'put on top of each other';




$u$ and $v$ adjacent: the edges joining them are shrunk down until $u$ and $v$ are identified.

Contraction may cause multiple edges to appear. These are ignored in graph colouring because any single edge joining two vertices is enough to require them to be coloured differently; extra edges give no extra information in this respect.

**Example** If contraction is applied to an edge of $K_3$ then the result, *ignoring multiple edges* is $K_2$:




**Theorem 2** Let $G$ be a simple graph.

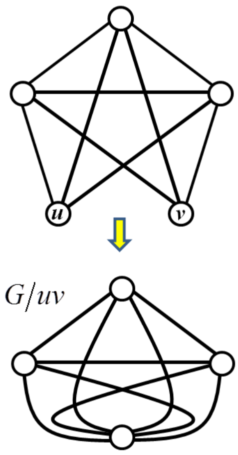
1. Suppose that $u$ and $v$ are not adjacent in $G$ and let $G + uv$ denote $G$ with edge $uv$ added. Then
$$C(G; \lambda) = C(G + uv; \lambda) + C(G/uv; \lambda). \tag{2}$$
2. Suppose that $e$ is an edge of $G$ and let $G\backslash e$ denote $G$ with edge $e$ deleted. Then
$$C(G; \lambda) = C(G\backslash e; \lambda) - C(G/e; \lambda). \tag{3}$$

**Proof.** (1) In any $\lambda$-colouring of $G$, either $u$ and $v$ are coloured differently or they are coloured the same. If differently then the colouring is still valid for $G + uv$; if the same then the colouring is a valid colouring for $G/uv$. So any $\lambda$-colouring of $G$ is either a $\lambda$-colouring of $G + uv$ or a $\lambda$-colouring of $G/uv$ but not both. So by the Summation Principle $C(G; \lambda) = C(G + uv; \lambda) + C(G/uv; \lambda)$.

(2) Making $u$ and $v$ the endpoints of $e$ we can put $G\backslash e$ in place of $G$ in equation (2) and rearrange to get equation (3). □

**Example** How many ways are there to 3-colour the graph $G$ on the right?

**Solution** Applying equation (2) of Theorem 2, the 3-colourings of the graph are precisely those when edge $uv$ is added and those when $u$ and $v$ are contracted. Now adding $uv$ gives $K_5$, and contracting $u$ and $v$ gives $K_4$, ignoring multiple edges (see the graph bottom right). So




$$\begin{aligned} C(G; \lambda) &= C(K_5; \lambda) + C(K_4; \lambda) \\ &= \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4) + (\lambda-1)(\lambda-2)(\lambda-3) \\ &= \lambda(\lambda-1)(\lambda-2)(\lambda-3)^2 \end{aligned}$$

So $C(G; \lambda) = 0$ for $\lambda \leq 3$ colours. This would be very easy to explain combinatorially! But the answer for $\lambda = 4$ colours would require more work; algebraically we have made it a trivial task: $C(G; 4) = 4 \times 3 \times 2 \times 1^3 = 24$.