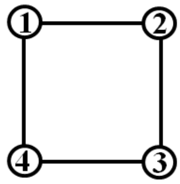
Chapter 9. Section 2

The chromatic polynomial

In the previous section we gave a combinatorial answer to the following question:

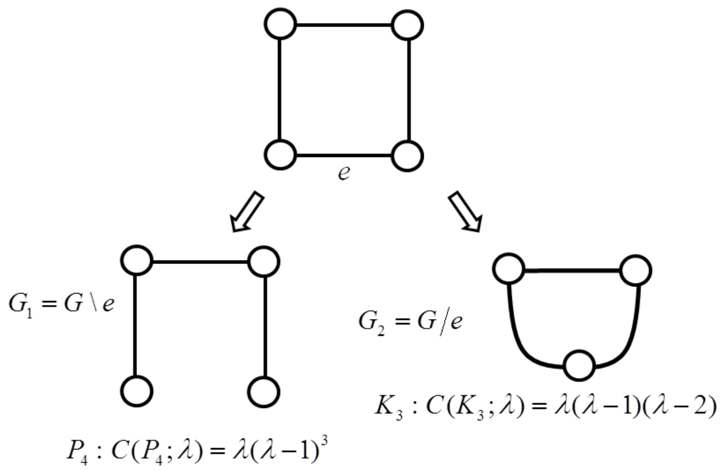
Example How many ways are there to colour the labelled copy of the 4-cycle C_4 on the right, using

(a) $\lambda = 2$ colours? (b) $\lambda = 3$ colours?



We are now able to calculate a purely algebraic answer using Theorem 2 from the previous lecture.

Here is the analysis: we apply deletion and contraction to any edge e. For both operations, we immediately get one of the special-case graphs from the last lecture, for which we can just write down the value of $C(G;\lambda)$. Now apply Theorem 2, equation (3):



$$\begin{aligned}
C(G;\lambda) &= C(P_4;\lambda) - C(K_3;\lambda)\\
&= \lambda(\lambda-1)^3 - \lambda(\lambda-1)(\lambda-2)\\
&= \lambda(\lambda-1)\left((\lambda-1)^2 - (\lambda-2)\right)\\
&= \lambda(\lambda-1)\left(\lambda^2 - 3\lambda + 3\right).
\end{aligned}$$

and the answer works for *any* λ:

$\lambda = 2$**:** $C(C_4;2) = 2(2-1)(4-6+3) = 2$; and

$\lambda = 3$**:** $C(C_4;3) = 3(3-1)(9-9+3) = 18$; confirming our previous combinatorial analysis;

$\lambda = 4$**:** $C(C_4;4) = 4(4-1)(16-12+3) = 84$; going beyond it — as far as we like!

Advantage: we do not have to rely on our intuition, nor on our ability to write down a complicated argument in words!

Disadvantage: we produce unlimited numerical answers on auto-pilot, without having to think!

Deletion and contraction are perfectly suited to proofs by induction on #edges, since both $G\setminus e$ and G/e have one fewer edges than G. Examples of easy properties of $C(G;\lambda)$ which can be proved in this way are given in the following corollary to Theorem 2, equation (3):

Corollary Let G be a simple graph with n vertices and m edges. Then

1. $C(G;\lambda)$ is a monic polynomial in λ (monic: coefficient of highest power = 1) whose coefficients alternate in sign $(+\ -\ +\ -\ \ldots)$.
2. The degree of this polynomial (i.e. highest power of λ) is equal to n.
3. The coefficient of λ^{n-1} is $-m$.
4. The lowest power of λ is equal to the number of connected components of G.

So $C(G;\lambda)$, if λ is taken to be a variable, is a polynomial! It can be evaluated at any real, or even complex, value of λ, although only the nonnegative integers can be interpreted in terms of numbers of colourings of G. This polynomial is called the **chromatic polynomial** of G and was originally introduced, almost exactly one hundred years ago, in the hope of proving the Four Colour Theorem. Because the Four Colour Theorem just says that $C(G;\lambda)$ does not have a root at $\lambda = 4$ for any planar graph G. Roots of polynomials have been studied for many hundreds of years — surely all this theory can be brought to bear on the Four Colour Theorem? In the end the answer is perhaps no, for reasons which have emerged from mathematical physics in the last 20 years. You can discover more at Theorem of the Day.