Chapter 5. Section 1

Generating Functions

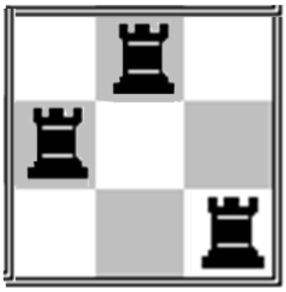
Generating functions are a way of manipulating lots of data as one single object. This is a similar idea to matrices: matrix arithmetic operates on whole arrays of numbers at the same time; generating function arithmetic operates on whole polynomials, or more generally whole power series, at the same time.

The **generating function** for a sequence $(a_n)_{n\geq 0}$ is the power series $\displaystyle\sum_{k=0}^{\infty} a_k x^k$.

Matrix arithmetic was invented in the 1850s and had to develop its own methods and rules. Power series have existed, in one form or another, for over a thousand years. So to import them into combinatorics, in the form of generating functions, means to import a huge collection of mathematical techniques and ideas!

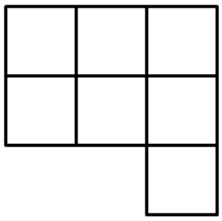
Rook Polynomials

In Week 1, we asked in how many ways might k non-attacking rooks be placed on an $n\times n$ chessboard. We gave the answer for $n = 3$ and $k = 0, 1, 2, 3$ in the form of a single polynomial: $1+9x+18x^2+6x^3$. The coefficient of x^3, for example, was 6 because there are $3! = 6$ permutations of [3] and these are in one-to-one correspondence with 3-rook placements. The permutation $1 \to 2, 2 \to 1, 3 \to 3$ is shown on the right.



A polynomial of finite degree d is a power series: just set to zero all coefficients beyond x^d. So polynomials are a special case of general generating functions.

Now suppose we have a chessboard in which some squares are disallowed. Rook placements on such a restricted board will correspond to partial functions $[n] \longrightarrow [n]$ with restrictions of the form $f(x) \neq x'$. For example, the chessboard on the right has two squares on its 3rd row disallowed.



$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \times & \times & \cdot \end{pmatrix} \end{array}$$

A placement of three rooks on this board will correspond to specifying a permutation in the matrix on the far right: 1 and 2 may be mapped to anything in [3] but 3 may only be mapped to 3. Questions about such permutations can be answered by manipulating rook polynomials.

For a chessboard C, denote by $R(C; x)$ the rook polynomial for C. So R is a polynomial function in x whose definition depends on your choice of chessboard C.

Theorem A Let C be chessboard with some disallowed squares. Let s be an allowed square.

Denote by $C\backslash s$ the **deletion** of s from C: s becomes a disallowed square.
Denote by C/s the **contraction** of s from C: the whole row and column of s become disallowed.

Then

$$R(C; x) = R(C\backslash s; x) + xR(C/s; x).$$

Let us apply this 'decomposition' to the restricted chessboard above: we get

$$\begin{aligned} R(\ ;x) &= R(\ ;x) + xR(\ ;x) \\ &= 1+6x+6x^2 + x(1+4x+2x^2) \qquad (1)\\ &= 1+7x+10x^2+2x^3. \qquad (2)\end{aligned}$$

(The $6x^2$ term on the left-hand polynomial in (1) is calculated directly: there are two rooks; one may occupy any of the three top-row positions; the other is then left with two possible positions remaining on the bottom row. Now apply the Product Principle.)

Theorem B Let C and D be two regions of a chessboard which have no rows or columns in common: they may be regarded as 'disjoint' chessboards. Let $C \cup D$ denote the chessboard whose allowed squares are precisely those of C or D. Then

$$R(C \cup D; x) = R(C; x)R(D; x).$$

The 'decomposition' applies as in the following example

$$\begin{aligned} R(\ ;x) &= R(\ ;x) \times R(\ ;x) \\ &= (1+6x+6x^2)\times(1+2x) \\ &= 1+8x+18x^2+12x^3. \qquad (3)\end{aligned}$$

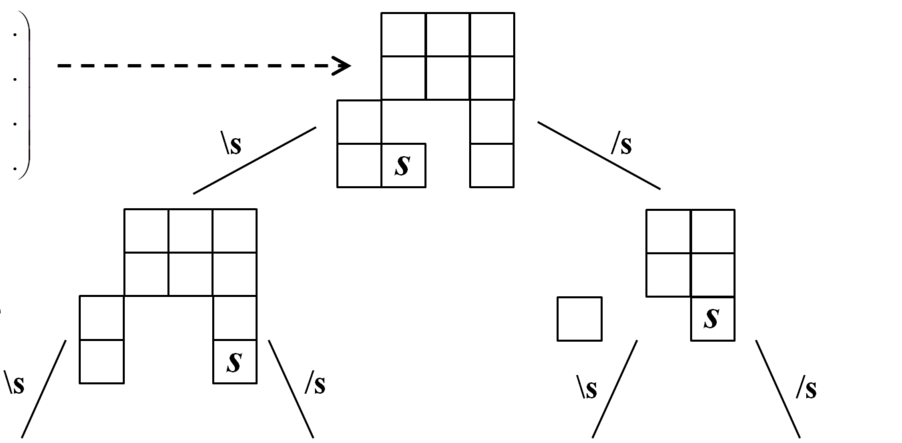
Now we apply Theorems A and B to an example question about restricted permutations:

Example How many permutations are possible on [4] if: only 3 or 4 may map to 1 and 2; 3 may not map to 2 or 3; 4 may not map to 3?

Solution We shall show, in the form of a 'tree', how Theorems A and B are applied to solve this.

The question is equivalent to asking for 4-rook placements in the restricted 'matrix' shown here on the right:

$$\begin{pmatrix} \times & . & . & . \\ \times & . & . & . \\ . & \times & \times & . \\ . & . & \times & . \end{pmatrix}$$

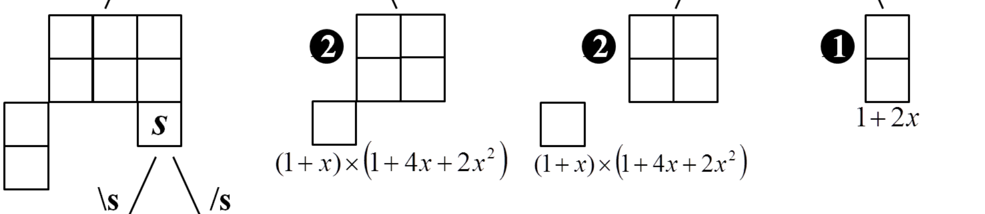


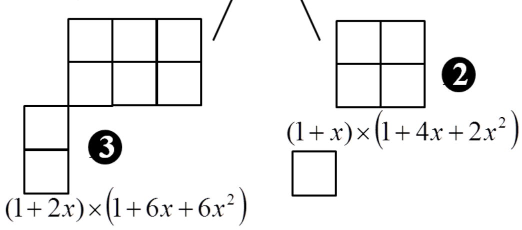
Selecting squares 's' repeatedly, we break down the restricted chessboard until we achieve boards for which we can write down the rook polynomials directly, or via Theorem B:

❶ $= 1+2x$;

❷ $= (1+x)(1+4x+2x^2)$;
$= 1+5x+6x^2+2x^3$

❸ $= (1+2x)(1+6x+6x^2)$
$= 1+8x+18x^2+12x^3$.





So far, only Theorem B has been applied, to calculate the polynomials of small chessboards...

... but now Theorem A is applied:

$$\begin{aligned} R(C;x) &= ((❸ + x❷) + x❷) + x(❷ + x❶) \\ &= 1+8x+18x^2+12x^3 \\ &\quad +3x(1+5x+6x^2+2x^3) \\ &\quad +x^2(1+2x) \\ &= 1+11x+34x^2+32x^3+6x^4. \end{aligned}$$

And we have our answer: there are 6 restricted permutations as counted by the coefficient of x^4.