Chapter 6. Section 1

Applying Generating Functions to Linear Recurrences

We will now justify, although only via a typical example, the solution of a linear homogeneous recurrence relation in terms of the roots of its characteristic equation (see Chapter 3, Section 3).

Example Solve $a_n = 2a_{n-1} + 3a_{n-2}$, $n \geq 2$, $a_0 = 1$, $a_1 = 2$.

Solution: We have an infinite sequence $(a_n)_{n\geq 0}$ which we can express as a single power series $\sum_{k=0}^{\infty} a_k x^k$ which is the **generating function** for the sequence.

Write $A(x)$ for this generating function. You can calculate the next three values: $a_2 = 7$, $a_3 = 20$, $a_4 = 61$, so

$$A(x) = 1 + 2x + 7x^2 + 20x^3 + 61x^3 + \ldots.$$

Solving the recurrence means giving a **closed form** for the generating function.

Rearranging the recurrence relation we can specify another generating function for the values $a_n - 2a_{n-1} - 3a_{n-2}$, for $n \geq 2$. If we call this $C(x) = \sum_{k=0}^{\infty} c_k x^k$ then we know that $c_2, c_3, \ldots$ are all zero because of the original recurrence. But we have to assign a value to c_0 and c_1.

Suppose we can find a *third* generating function $D(x) = d_0 + d_1 x + d_2 x^2 + \ldots = \sum_{k=0}^{\infty} d_k x^k$, which will multiply with $A(x)$ to give $C(x)$. It will turn out this works for just one unique choice of c_0 and c_1, just what we were looking for! Here we go:

$$\begin{aligned} A(x)D(x) = C(x) &= \sum_{k=0}^{\infty} c_k x^k & (1)\\ &= (a_0 + a_1 x + a_2 x^2 + \ldots)(d_0 + d_1 x + d_2 x^2 + \ldots) \\ &= a_0 d_0 + (a_0 d_1 + a_1 d_0)x + (a_0 d_2 + a_1 d_1 + a_2 d_0)x^2 + \ldots. & (2) \end{aligned}$$

Now, comparing (1) and (2) we see that

$$c_n = a_0 d_n + a_1 d_{n-1} + \ldots + a_{n-2} d_2 + a_{n-1} d_1 + a_n d_0,$$

but $c_n = a_n - 2a_{n-1} - 3a_{n-2}$ and comparing coefficients, we must have:

$$\begin{array}{rll} a_n = a_n d_0, \quad \text{so} \quad d_0 = 1, & \text{which means} & c_0 = a_0 d_0 = 1 \times 1 = 1 \\ -2a_{n-1} = a_{n-1} d_1, \quad \text{so} \quad d_1 = -2, & \text{which means} & c_1 = a_0 d_1 + a_1 d_0 = 1 \times -2 + 2 \times 1 = 0 \\ -3a_{n-2} = a_{n-2} d_2, \quad \text{so} \quad d_2 = -3, & \text{which } \textit{should} \text{ mean} & c_2 = 0 \ldots \\ \text{(and now all further } d_n \text{ zero)} & \text{...and indeed} & a_0 d_2 + a_1 d_1 + a_2 d_0 = 1 \times -3 + 2 \times -2 + 7 \times 1 = 0. \end{array}$$

(The last equality *had* to happen because we supposed that $c_2 = a_2 - 2a_1 - 3a_0$ which we already know is zero.)

So now we can go back to equation (1) and write it out in full:

$$\begin{aligned} A(x)(d_0 + d_1 x + d_2 x^2 + d_3 x^3 + \ldots) &= c_0 + c_1 x + c_2 x^2 + \ldots \\ \text{becomes:} \quad A(x)(1 - 2x - 3x^2) &= 1 + 0x + 0x^2 + 0x^3 + \ldots \\ \text{or:} \quad A(x)(1 - 2x - 3x^2) &= 1 \quad \text{and...} & (3) \\ \text{... we have our closed form:} \quad A(x) &= \frac{1}{1 - 2x - 3x^2}. & (4) \end{aligned}$$

Our closed form solution $A(x) = (1 - 2x - 3x^2)^{-1}$ says that $A(x)$ is the reciprocal of the characteristic equation of the recurrence! Still, our solution in Week 3 gave us a closed form for a_n, rather than for the generating function for the whole recurrence. And we can get this too, via partial fractions separation and the binomial theorem:

$$\begin{aligned}
\frac{1}{1-2x-3x^2} &= \frac{1}{(1-3x)(1+x)} \\
&= \frac{3}{4(1-3x)} + \frac{1}{4(1+x)} \\
&= \frac{3}{4}\left(1 + 3x + (3x)^2 + (3x)^3 + \ldots\right) + \frac{1}{4}\left(1 - x + x^2 - x^3 + \ldots\right) \quad \text{(binomial theorem)} \\
&= \frac{1}{4}\left(3\sum_{k=0}^{\infty} 3^k x^k + \sum_{k=0}^{\infty}(-1)^k x^k\right) \\
&= \frac{1}{4}\sum_{k=0}^{\infty}\left(3^{k+1} + (-1)^k\right)x^k.
\end{aligned}$$

This is our generating function $A(x)$. So the n-th term is $a_n = \frac{1}{4}\left(3^{n+1} + (-1)^n\right)$.

The partial fractions separation reveals why a characteristic equation with repeated roots introduces a factor of n into the solution. We will need to calculate another binomial theorem expansion:

$$\frac{1}{(1+x)^2} = (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots = \sum_{k=0}^{\infty}(-1)^k(k+1)x^k \tag{5}$$

Example Use the method of generating functions to solve $a_n = 4a_{n-1} - 4a_{n-2}$, $n \geq 2$ and $a_0 = 1, a_1 = 3$.

Solution: We want a closed form for $A(x)$ which we can calculate to be:

$$A(x) = 1 + 3x + 8x^2 + 20x^3 + 48x^4 + \ldots.$$

Following the same procedure we will create $A(x)D(x) = C(x)$ to satisfy $c_n = a_n - 4a_{n-1} + 4a_{n-2}$ and comparing coefficients, we must have:

$$\begin{array}{rll}
a_n = a_n d_0, \quad \text{so} \quad d_0 = 1, & \text{which means} & c_0 = a_0 d_0 = 1 \times 1 = 1 \\
-4a_{n-1} = a_{n-1} d_1, \quad \text{so} \quad d_1 = -4, & \text{which means} & c_1 = a_0 d_1 + a_1 d_0 = 1 \times -4 + 3 \times 1 = -1 \\
4a_{n-2} = a_{n-2} d_2, \quad \text{so} \quad d_2 = 4, & \text{which } \textit{should} \text{ mean} & c_2 = 0 \\
& \text{and indeed} & a_0 d_2 + a_1 d_1 + a_2 d_0 = 1 \times 4 + 3 \times -4 + 8 \times 1 = 0.
\end{array}$$

Then we get $A(x)(1 - 4x + 4x^2) = 1 - x$ so

$$\begin{aligned}
A(x) &= \frac{1-x}{1-4x+4x^2} = \frac{1-x}{(1-2x)^2} \\
&= \frac{1}{2(1-2x)} + \frac{1}{2(1-2x)^2} \quad \text{(partial fractions separation)} \\
&= \frac{1}{2}\left(\sum_{k=0}^{\infty} 2^k x^k + \sum_{k=0}^{\infty}(k+1)2^k x^k\right) \quad \text{(binomal theorem, including equation (5))} \\
&= \frac{1}{2}\sum_{k=0}^{\infty}\left(2(2^k) + k2^k\right)x^k = \sum_{k=0}^{\infty}\left(2^k + k2^{k-1}\right)x^k.
\end{aligned}$$

This is our generating function $A(x)$. So the n-th term is $a_n = 2^n + n2^{n-1}$, $n \geq 0$.