Chapter 6. Section 2

An Example Exam Question

The following is Question 3 from the 2011 exam paper.

Example

(a) Explain what is meant by the generating function for the series $a_0, a_1, a_2, \ldots$.

(b) Compute a closed formula for the generating function of the series defined by $a_n = n$, for $n \geq 0$.

(c) Compute a closed formula for the generating function of the series defined by $a_n = 2a_{n-1} - a_{n-2}$, for $n \geq 2$, and the initial conditions $a_0 = 0, a_1 = 2$.

Solution:

(a) Just writing $A(x) = \sum_{k=0}^{\infty} a_k x^k$ ought to get full marks (2 out of the total of 12).

(b) The generating function is $\sum_{k=0}^{\infty} kx^k = x + 2x^2 + 3x^3 + 4x^4 + \ldots$.

Now we met something similar in the last lecture: $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$. This is nearly right, but we are out by a multiple of x:

$$x(1+x)^{-2} = x - 2x^2 + 3x^3 - 4x^4 + \ldots.$$

And we are out by a multiple of $(-1)^k$. We recognise that this is precisely the difference between $(1+x)^n$ and $(1-x)^n$ for any power n. So our answer (for another 4 marks) is

$$\sum_{k=0}^{\infty} kx^k = x(1-x)^{-2}.$$

(c) The procedure was to hypothesis a product $A(x)D(x) = C(x)$ where $C(x)$ is the generating function with $c_n = a_n - 2a_{n-1} + a_{n-2}$. We will find that $D(x) = 1 - 2x + x^2$. And $c_n = 0$ by definition for $n \geq 2$, while for $n = 0$ we have $c_0 = a_0 d_0 = 0 \times 1 = 0$ and for $n = 1$ we have $c_1 = a_0 d_1 + a_1 d_0 = 0 \times -2 + 2 \times 1 = 2$.

Now $A(x)(1 - 2x + x^2) = 2x$ so $A(x) = \dfrac{2x}{1 - 2x + x^2}$.

This earns the final 6 marks out of 12. The question only asks for a closed formula for the generating function, rather than for a_n. However, we know we can do that too. Having answered part (b) we realise we don't even need to bother with partial fractions:

$$\begin{aligned} A(x) &= \frac{2x}{1 - 2x + x^2} = \frac{2x}{(1-x)^2} = 2x(1 + 2x + 3x^2 + 4x^4 + \ldots) = 2x + 4x^2 + 6x^3 + 8x^4 + \ldots \\ &= \sum_{k=0}^{\infty} 2kx^k. \end{aligned}$$

Sure enough, if we calculate a few terms using the original recurrence we find we get exactly the nonnegative even numbers!