Chapter 6. Section 3

Exponential Generating Functions

We want to arrange a set of n objects into some structure.

Examples

1. Arrange $[5] = \{1, 2, 3, 4, 5\}$ as a set. This is **trivial**: it is already a set, so you don't have to do anything. There is only one way to do this.
2. Arrange [5] into two nonempty subsets. You get $\{1\}, \{2, 3, 4, 5\}$ or $\{2\}, \{1, 3, 4, 5\}$, or $\{1, 2\}, \{3, 4, 5\}$, etc: i.e. you get all **2-partitions** of [5]. The number of ways of partitioning $[n]$ into k nonempty subsets is denoted by $S(n, k)$, these numbers being called the **Stirling numbers of the second kind**. There are $S(n, 2) = 2^{n-1} - 1$ ways to make our arrangement, for $n \geq 1$ (can you think of a combinatorial argument for this?)
3. Arrange $[n]$ into an ordered list. You get $1, 2, 3, 4, 5$, or $1, 3, 2, 4, 5$ or $1, 3, 4, 2, 5$, etc: i.e. all permutations of $[n]$. There are $n!$ ways to do this.

Counting these kinds of arrangements can often be done very elegantly using a type of generating function which differs from the **ordinary generating functions** we have used so far in that *the k-th term is divided by* $k!$. It seems like a superficial and messy thing to do ... but it works!

If $(a_n)_{n\geq 0}$ is a sequence then the **exponential generating function** (abbreviated to **egf**) for the sequence is:

$$A(x) = \sum_{k=0}^{\infty} a_k \frac{x^k}{k!}.$$

Examples

1. The number of ways of arranging $[n]$ (trivially) as a set, for $n \geq 0$, has egf:

$$A(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = e^x. \tag{1}$$

2. The number of ways of arranging $[n]$ (still trivially) as a *nonempty* set, for $n \geq 0$, has egf:

$$A(x) = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = e^x - 1. \tag{2}$$

3. The number of ways of arranging [2] (trivially) as an $[n]$-element set, $n \geq 0$ has egf:

$$A(x) = \frac{x^2}{2!} \tag{3}$$

since there are zero possibilities for every n except $n = 2$.

4. The number of 2-partitions of $[n]$, for $n \geq 0$ has egf:

$$\begin{aligned} A(x) = \sum_{k=0}^{\infty} S(n,2)\frac{x^k}{k!} &= \frac{x^2}{2!} + \frac{3x^3}{3!} + \frac{7x^4}{4!} + \dots \\ &= \sum_{k=1}^{\infty}\left(2^{k-1} - 1\right)\frac{x^k}{k!} \\ &= \frac{1}{2}\sum_{k=1}^{\infty} 2^k \frac{x^k}{k!} - \sum_{k=1}^{\infty}\frac{x^k}{k!} \\ &= \frac{1}{2}\left(\sum_{k=0}^{\infty}\frac{(2x)^k}{k!} - 1\right) - \left(\sum_{k=0}^{\infty}\frac{x^k}{k!} - 1\right) \\ &= \frac{1}{2}\left(e^{2x} - 1\right) - e^x + 1 = \frac{1}{2}\left(e^x - 1\right)^2. \end{aligned} \tag{4}$$

5. The number of ways to arrange $[n]$ into an ordered list (i.e. permutation) has egf:

$$A(x) = \sum_{k=0}^{\infty} k!\frac{x^k}{k!} = 1 + x + x^2 + \dots = \frac{1}{1-x} \tag{5}$$

   which is the *ordinary* generating function for arranging $[n]$ as an (unordered) set. Egf's are often described as 'replacing unlabelled objects with labelled objects', and here we see an example: $\{1,2,3,4,5\}$ and $\{2,4,1,5,3\}$ are the same set; but if we label positions: 1st position, 2nd position, etc, then they become *different* permutations.

6. The number of ways to arrange $[n]$ (trivially) as a set of even cardinality has egf:

$$A(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

   since there is one way to make the arrangement if n is even and zero ways if n is odd. This gives us the hyperbolic cosine of x:

$$\begin{aligned} A(x) &= \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \right. \\ &\qquad \left. +1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots\right) = \frac{1}{2}\left(e^x - e^{-x}\right) = \cosh x. \end{aligned} \tag{6}$$

The power of egf's comes from:

The Product Principle for EGF's: suppose we have two independent ways of arranging sets (into structures), say A_1 and A_2. Use $A_1 \cup A_2$ to denote the arrangements which consist of:

1. Dividing a set into two subsets;
2. Applying A_1 to one subset and apply A_2 (independently) to the other.

Let $A_1(x)$ and $A_2(x)$ be the egf's for A_1 arrangements and A_2 arrangements. Let $A_1 \cup A_2(x)$ be the egf for $A_1 \cup A_2$ arrangements. Then

$$A_1 \cup A_2(x) = A_1(x)A_2(x). \tag{7}$$

We can extend this inductively to more than two sets: $A_1 \cup A_2 \cup \dots \cup A_n(x) = A_1(x)A_2(x)\cdots A_n(x)$.

Examples

1. For the egf for the number of subsets of $[n]$, take A_1 to be the trivial subset arrangement with egf e^x: this will record the elements which are *in* are subset. Take A_2 to be the trivial subset arrangement for the other half of the partition of $[n]$, which will be elements which are *not in* the subset. Then

$$A_1 \cup A_2(x) = e^x \times e^x = e^{2x} = \sum \frac{(2x)^k}{k!} = \sum 2^k \frac{x^k}{k!},$$

so #subsets of $[n]$ is $a_n = 2^n$.

2. The number of nonempty subsets of $[n]$ will be given, similarly, by $(e^x - 1) \times e^x$, since A_1 must arrange a nonempty subset of 'in' elements. This gives $A_1 \cup A_2(x) = \sum 2^k \frac{x^k}{k!} - \sum \frac{x^k}{k!}$ with $a_n = 2^n - 1$.

3. For the number of functions $[n] \to [N]$, for a given value of N, we partition $[n]$ into N possibly empty subsets: elements of $[n]$ mapped to 1, those mapped to 2, . . ., those mapped to N. Then the egf counting functions is $(e^x)^N = \sum \frac{(Nx)^k}{k!} = \sum N^k \frac{x^k}{k!}$. And so #functions $[n] \to [N]$ is given as $a_n = N^n$.

4. Similarly, #*surjective* functions $[n] \to [N]$ has egf $(e^x - 1)^N = \sum_{i=0}^{N} \binom{N}{i} (-1)^i e^{N-i}$. From here we can derive: #surjective functions $[n] \to [N] = \sum_{i=0}^{N-1} (-1)^i \binom{N}{i} (N-i)^k$ (Chapter 3, Section 1).