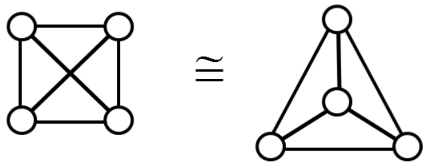
Chapter 8. Section 1

Planarity

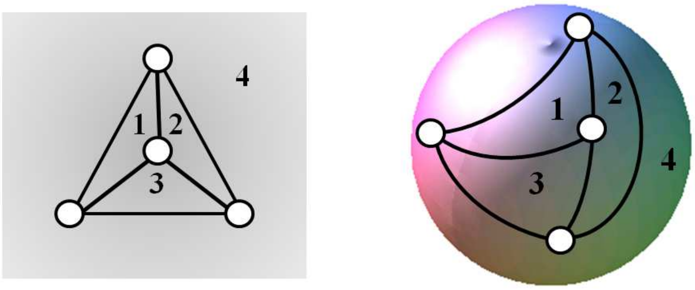
We saw in Chapter 7 Lecture 1 that K_4, the complete graph on 4 vertices, could be drawn so that no edges crossed.



With a little more formality we can that a graph drawing is an **embedding** in the plane, or equivalently on the surface of a sphere, in such a way that vertices are mapped to distinct points and edges are mapped to continuous curves. A **plane** drawing is one in which any two distinct edges map to curves which are disjoint except at points corresponding to vertices incident with both edges. We do not want to get into topological details, however.

A graph which admits a plane drawing is called **planar**. Thus K_4 is a planar graph (whether or not a particular drawing is a plane drawing); in fact K_n is planar for all $n \leq 4$. But K_5 is not planar, as we shall see.

A plane drawing of a graph partitions the plane (or surface of the sphere) into disjoint regions called **faces**, each bounded by the curves which map some subset of the edges. Below are two drawings of K_4: the left-hand drawing is on the plane, with one face being 'outside'; the right-hand drawing is on the sphere and all faces are 'inside' in the sense that if we walk in a straight line (or more correctly **great circle**) from a point inside any face, even face no. 4, we will eventually meet an edge of the graph.

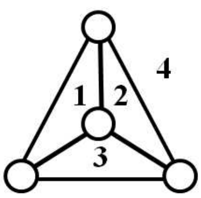


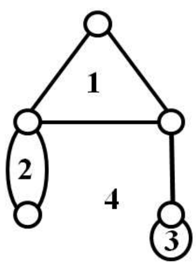
A classic 1750 theorem of Leonhard Euler shows that the number of faces in any plane drawing of a graph is necessarily related to the number of edges and vertices of the graph:

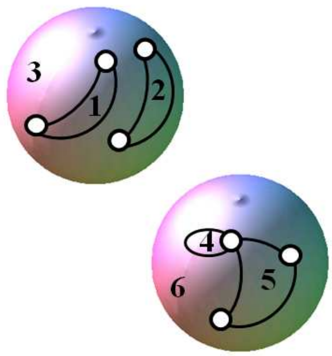
Theorem For any plane drawing of a connected graph having n vertices, e edges and f faces,

$$n - e + f = 2.$$

Examples







(1) $n = 4$, $e = 6$, $f = 4$,
$n - e + f = 4 - 6 + 4 = 2$.

(2) $n = 5$, $e = 7$, $f = 4$,
$n - e + f = 5 - 7 + 4 = 2$.

(3) $n = 7$, $e = 8$, $f = 6$,
$n - e + f = 7 - 8 + 6 = 5$.

Example 3 above seems like a counterexample! But it is not, because (a) the graph is not connected: it has three **connected components** (two on the first sphere and one on the second); and (b) it is not embedded in a single sphere. We can generalise Euler's theorem to say:

$$n - e + f = \#\text{components} + \#\text{spheres},$$

which makes example 3 correct and also suggests why Euler's equation has a '2' on the right-hand side.[1]

Although the spheres and components idea seems to give a definitive explanation of the '2' in Euler's theorem, we will find another explanation in the following proof of the original theorem:

Proof of Euler's Theorem Let G be the graph and let E be the collection of edges of G.

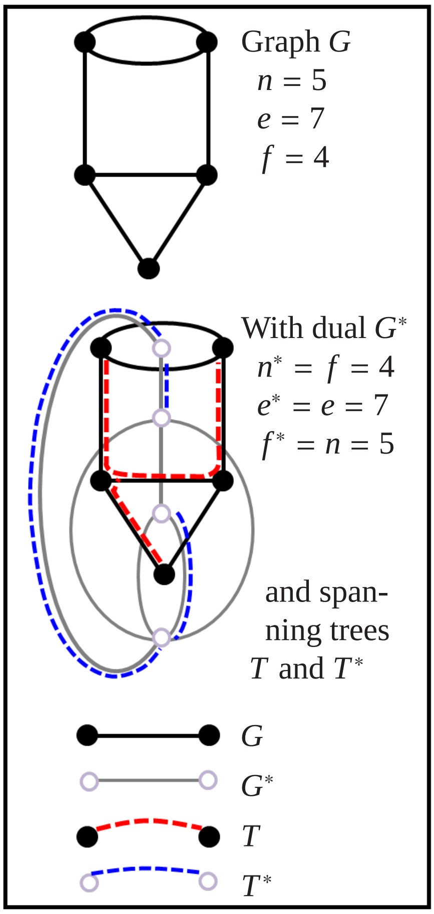
The steps of the proof are accompanied by an example on the right;[2]the example is not part of the proof, of course, but probably suffices to fill in a few missing details!

1. Take a spanning tree T of G. Then T contains all n vertices, because it is spanning; so it has $n-1$ edges because any tree on n vertices has $n-1$ edges.

2. Construct the **dual graph** of G. The dual graph has a vertex corresponding to each of the f faces. For each pair, say u and v, of these dual graph vertices there is a distinct edge joining them for every edge of G separating the faces corresponding to u and v.

3. It can be shown that the edges of the dual graph G^* that do not cross edges of T form a spanning tree T^* of G^*. Since G^* has f vertices T^* has $f-1$ edges.

4. All edges in G are either in T or are crossed by edges of T^*. So $e = (n-1) + (f-1)$ and this rearranges to give $n - e + f = 2$. □



So now the '2' in Euler's theorem comes from two deficits: a -1 for each of the spanning trees 'living in' the planar drawing of G.

Corollary 1 Let G be a simple, planar graph with n vertices and e edges. Then $e \leq 3n-6$.

Proof. Embed G in the plane. Suppose the embedding has f faces. Now we double count faces and edges. Each face has at least 3 edges, so if we add all the edges on all the faces we get at least $3f$ edges. But during this count we see every edge exactly twice, since every edge belongs to exactly two faces. So we have $2e \geq 3f$, or

$$\begin{aligned}
-f &\geq -\frac{2}{3}e. && (1)\\
\text{Euler's theorem:}\quad n-e+f &= 2 \\
\text{or}\quad n-e-2 &= -f. && (2)\\
(1)\ \&\ (2):\ n-e-2 &\geq -\frac{2}{3}e \\
\text{so}\quad 3n-3e-6 &\geq -2e \\
\text{or}\quad 3n-6 &\geq e
\end{aligned}$$

In the next lecture we see how Corollary 1, and a slight variation of it, help to determine whether or not a given graph is planar.

[1]And that is not the end of the story: around 1960 Imre Lakatos did a whole PhD in the philosophy of mathematics based on finding successive counterexamples to Euler's theorem and then 'repairing' it by successive generalisations. This became a classic book: *Proofs and Refutations. The Logic of Mathematical Discovery*.

[2]Reproduced from Theorem of the Day.