Chapter 8. Section 2

Kuratowski's Theorem

At the end of Chapter 8, Section 1 we proved the following corollary to Euler's $n - e + f = 2$ theorem:

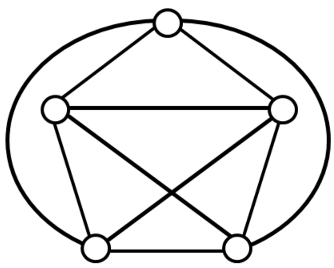
Corollary 1 Let G be a simple, planar graph with n vertices and e edges. Then $e \leq 3n - 6$.

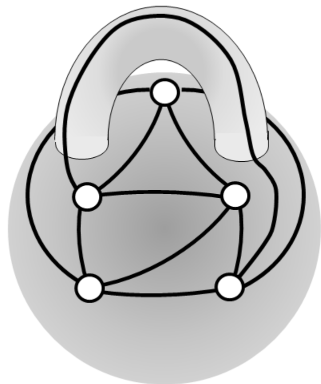
This immediately reveals that:

Corollary 2 K_5 is nonplanar.

Proof. K_5 has $n = 5$ vertices and $e = \binom{5}{2} = 10$ edges. But $3n - 6 = 9 < e$ so K_5 does not satisfy Corollary 1 and therefore cannot be planar. $\square$

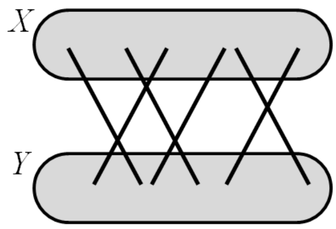
We can draw K_5 with just 1 crossing as shown below left. Corollary 2 shows that a crossing cannot be avoided, no matter how we embed K_5 in the plane or sphere. (However, if we embed in the torus, topologically equivalent to a sphere with a 'handle', then we *can* avoid crossings! This is shown below right.)



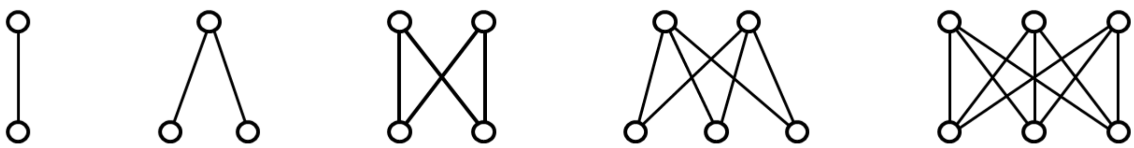


We will find one more nonplanar graph and together with K_5 this will turn out to be effectively a complete catalogue!

A **bipartite graph** is one whose vertex set V can be partitioned into two subsets X and Y so that all edge join vertices in X to vertices in Y (no adjacencies exist *within* X or Y. This property can be viewed diagrammatically as shown on the right. The sets X and Y are called the **bipartition** of V.

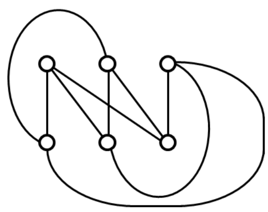


The **complete bipartite graph of order** m, n, denoted by $K_{m,n}$, is the bipartite graph with bipartition X of size m and Y of size n and all possible edges between X and Y. There are $n \times m$ edges in $K_{m,n}$. Some small complete bipartite graphs are shown below; the first four are planar although only the first is a plane drawing:



$K_{1,1} \cong K_2$ $K_{1,2} \cong P_3$ $K_{2,2} \cong C_4$ $K_{2,3}$ $K_{3,3}$

Now $K_{3,3}$ has $n = 6$ vertices and $e = 9$ edges and $3n - 6 = 12 \geq 9$ so Corollary 1 does not preclude the possibility of a plane drawing of $K_{3,3}$. As with K_5 we can quite easily get the number of crossings down to 1, as shown on the right. Nevertheless:

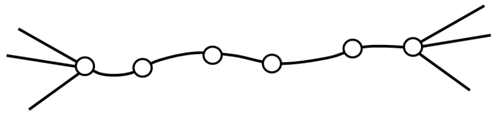


Corollary 3 $K_{3,3}$ is nonplanar.

Proof. Observe that no cycle in a bipartite graph can have length 3 (or indeed any odd length). So in any plane drawing of $K_{3,3}$, every face would have at least 4 edges. Then in the proof of Corollary 1 we can replace $2e \geq 3f$ with $2e \geq 4f$, giving $n - e - 2 \geq -\frac{2}{4}e = -\frac{1}{2}e$. The result is that, for any planar bipartite graph with n vertices and e edges, we must have $e \leq 2n - 4$. Now for $K_{3,3}$, we have $2n - 4 = 12 - 4 = 8 \ngeq 9$ contradicting this condition. So $K_{3,3}$ cannot have a plane drawing.

□

Suppose in a graph G we find a subgraph which is isomorphic to one of K_5 or $K_{3,3}$ except that some edges may be **subdivided** by adding vertices of degree two, as shown on the right.

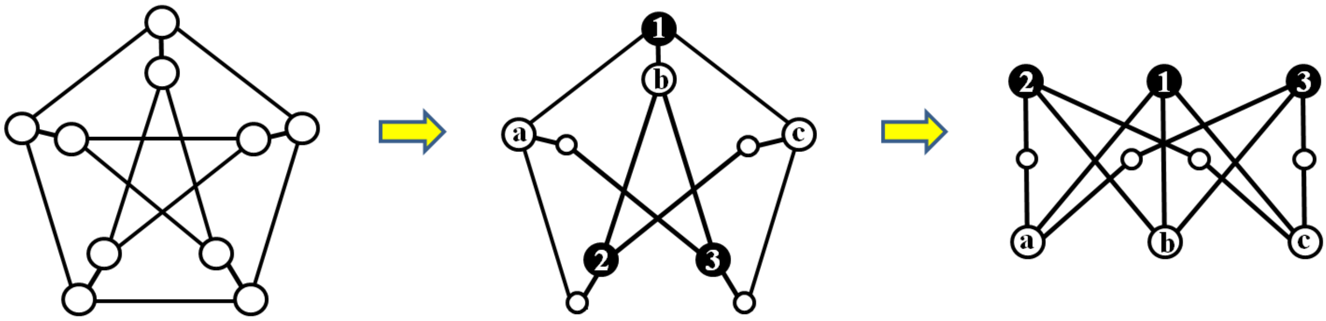


This is obviously irrelevant to planarity: a plane drawing will still be plane if we subdivide edges; a non-plane drawing cannot become plane thereby. So we have proved the **only if** part of one of graph theory's most famous theorems:

Theorem (Kuratowksi, 1930) A graph is planar if and only if it has no subgraph which is isomorphic to K_5 or $K_{3,3}$ with subdivided edges.

We omit the proof of the 'if' part of the theorem.

As an example of Kuratowski's theorem in action, we show that the Petersen graph (see Chapter 7, Section 2) is nonplanar. Even though it looks rather like K_5, it is actually a $K_{3,3}$ which we discover inside it:



The Petersen graph; a subgraph of it, which is a subdivided $K_{3,3}$.