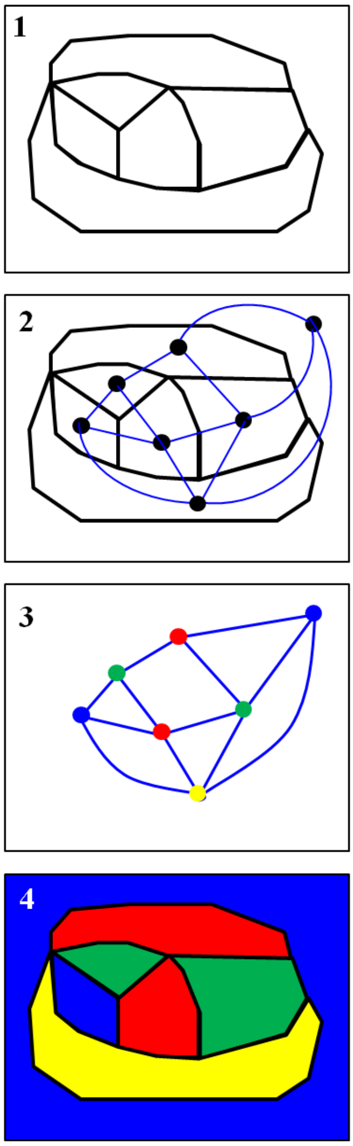
Chapter 8. Section 3



Colouring

Graph colouring is another major area within graph theory. It has its origins in a problem about planar graphs which we will not study in detail but which is an attractive example of mathematical modelling. It is illustrated on the right.

1. We have a map which we wish to colour with as few colours as possible so that no neighbouring countries are coloured the same. A famous problem from the 1850s asked whether four colours was always enough.

2. Treating map regions as planar graph faces and creating a 'dual' graph (as in the proof of Euler's theorem in Section 2) we turn this into a question about graphs: can the vertices of a planar graph be coloured with four colours so that no adjacent vertices are coloured the same?

3. The **Four Colour Theorem** whose proof, finally announced in 1976, controversially relied on computer processing of hundreds of special case 'configurations', says we can.

4. And, in the traditional manner of mathematical modelling, we map the solution in our model (the planar graph) back to our 'real-world' problem. Graph colouring has many much more serious applications, including timetabling and mobile phone frequency assignment, but we shall not cover these here. More details about the Four Colour Theorem can be found at Theorem of the Day.

The Four Colour Theorem is **best possible**, meaning we do sometimes need all four colours. This may be confirmed, for example, for the graph at (3), above right.

Let G be a simple graph. A **vertex colouring** of G is an assignment of colours to the vertices of G in such a way that no adjacent vertices are assigned the same colour. The **chromatic number** of G, denoted by $\chi(G)$, is the smallest number of colours for which this is possible.

Examples

1. $\chi(K_2) = 2$: any graph that contains at least one edge (isomorphic to K_2) requires at least two colours, one for each endpoint.
2. $\chi(K_3) = 3$: any graph that contains a triangle requires at least three colours (observe the four occurrences of K_3 in the example graph at (3) above right).
3. $\chi(K_n) = n$: consider (1) or (2) as the base cases for an induction; assume K_N requires N colours; obtain K_{N+1} by adding a vertex adjacent to every vertex of K_N, which must therefore be coloured differently from all these N vertices, hence $\chi(K_{N+1}) = N + 1$.
4. $\chi(G) = 2$ for any bipartite graph G: by definition all edges join X-vertices to Y-vertices; colour X red and Y blue.
5. $\chi(C_n) = 2$ for even n, and $= 3$ for odd n: C_{2k} is bipartite; construct C_{2k+1} by **subdividing** an edge of C_{2k}; the new vertex is adjacent to red and blue vertices and must therefore be green.
6. For any graph G, $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G)$ denotes the maximum vertex degree of G. Equality occurs if and only if $G = K_n$ or $G = C_{2k+1}$. This is **Brooks' Theorem** (1941).
7. The Four Colour Theorem: $\chi(G) \leq 4$ for any planar graph G.