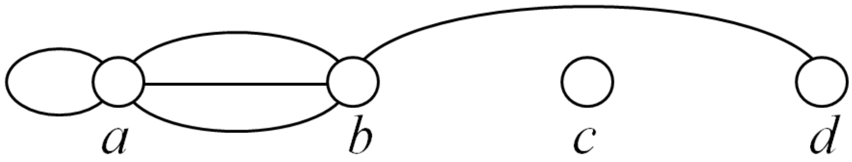
Chapter 1. Section 2

Graph Theory

Graphs



A **graph** G is a pair (V, E) where V is a finite set and E is a collection of subsets of V of size at most 2.

Example: The above picture is a pictorial representation of the following graph:

$$\begin{aligned} V &= \{a, b, c, d\}, \\ E &= \big\{\{a\}, \{a, b\}, \{a, b\}, \{a, b\}, \{b, d\}\big\} \end{aligned} \tag{1}$$

Elements of V are called **vertices**.

Elements of E are called **edges**.

Elements of elements of E (the vertices of the sets of the collection) are called **endpoints**.

Example: In (1) the edge $\{a\}$ has one endpoint a; the edge $\{b, d\}$ has endpoints b and d.

Usually it is easier to write edges as pairs of endpoints; (1) becomes:

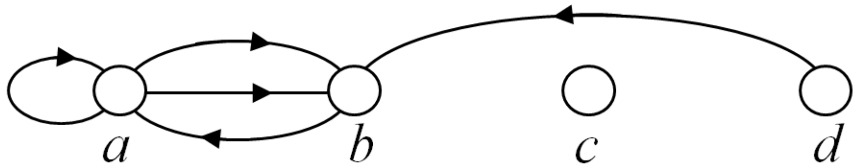
$$\begin{aligned} V &= \{a, b, c, d\}, \\ E &= \big\{aa,\ ab,\ ab,\ ab,\ bd\big\} \end{aligned}$$

Subsets of size 1 in E are called **self-loops**. In (1), $\{a\}$ is a self-loop.

Subsets in E which are repeated are called **multiple edges**. In (1), $\{a, b\}$ is an edge of multiplicity 3.

Graphs with neither self-loops nor multiple edges are called **simple**. In this case E is a *set* (rather than a collection) of sets of size exactly 2.

Digraphs



If we replace the word 'set' in the definition of the edge set by 'ordered pair' then we have a **directed graph** or **digraph**.

Example: The above picture is a pictorial representation of the following digraph:

$$\begin{aligned} V &= \{a, b, c, d\}, \\ E &= \big\{(a, a), (a, b), (a, b), (b, a), (d, b)\big\} \end{aligned} \tag{2}$$

Again we can simplify the notation for the collection E to give $E = \{aa, ab, ab, ba, db\}$. Notice that the order in which we list the endpoints is now important: $ab \neq ba$.

The following definitions apply to graphs or digraphs:

Two vertices are **adjacent** if they appear in the same subset of E. They are said to be **joined** by this edge.

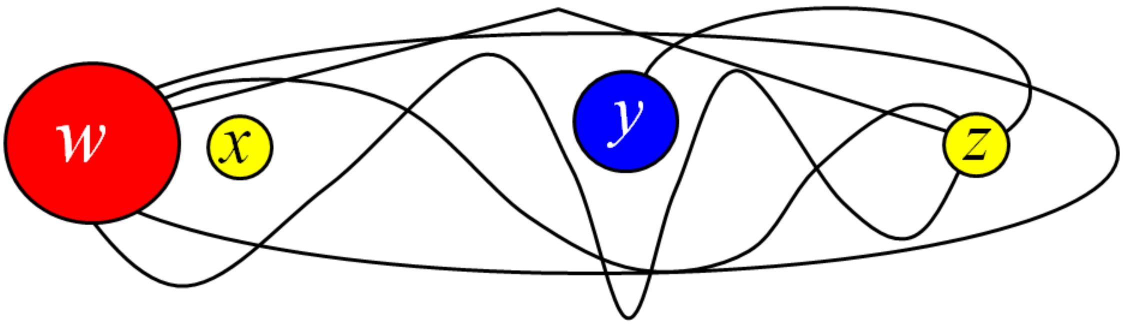
An edge and its endpoints are said to be **incident** with each other. For example, in (1) or (2) a is incident with ab (three times for the graph; twice for the digraph); and bd is incident with b and with d.

The number of edges incident with a vertex v is the **degree** of v, denoted by $\mathrm{d}(v)$. Self-loops count twice. In (1) or (2) we have $\mathrm{d}(a) = 5$, $\mathrm{d}(b) = 4$, $\mathrm{d}(c) = 0$ and $\mathrm{d}(d) = 1$.

A vertex of degree zero is said to be **isolated**.

For digraphs we distinguish between 'indegree' and 'outdegree' but we defer discussion until Chapter 5.

In many cases a pictorial representation of a graph or digraph (as used above) is just as good as the mathematical notation of sets and subsets. However, it is important to realise that this picture is not unique. For example, the following picture represents the same structure of vertex-edge incidences as the graph defined in (1):



The graph represented here is not identical to (1) only in that the vertex names are different. The way in which the edges cross over each other may or may not be of interest (perhaps the graph is model of a road network); similarly the colours on the vertices may be important (perhaps they represent different countries). And the size of the vertices may mean something (populations?) However, these details are **not** part of the mathematical object which is the graph G.