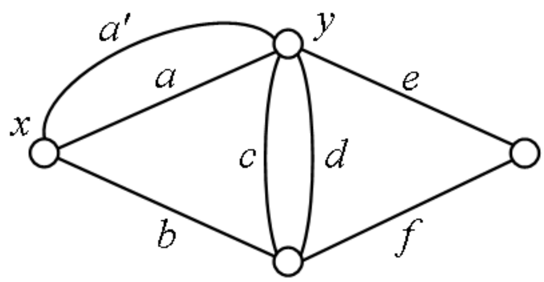
Chapter 10. Section 2

The Euler-Hierholzer Theorem

Here is the example graph from the last lecture but with an extra edge a' added:



It is not hard to see that an Euler tour is no longer possible: indeed, suppose there *was* a tour, S, say. We can assume, without loss of generality[1] that S starts with edge a (if it starts with b and ends with a we can just reverse the tour; and a and a' are copies of the same edge). If S ends with a' then how can edge b be included in the tour? If S ends with b then how can a' be included? We conclude that the number of edges at x is incompatible with an Euler tour.

The argument above that a vertex of degree 3 does not allow an Euler tour can be extended to a necessary and sufficient condition:

The Euler-Hierholzer Theorem Theorem of the Day A graph G is Eulerian if and only if it is connected and every vertex of G has even degree.

Proof. We first prove the necessity (the 'only if' or $\Rightarrow$ part): firstly, G must obviously be connected in order for a closed trail to pass every edge. For the degree condition, assume there is an Euler tour, S, say, and let v be any vertex. We consider two cases:

I: S begins and ends at v. Then the first and last edges of S are adjacent to v. Suppose S makes an additional k visits to v; each visit arrives at v via a new edge and leaves v via another new edge. Since every edge of G appears in S, by definition, we calculate the degree of v as

$$d(v) = 1 + 1 + 2 \times k,$$

which is even.

II: S does not begin and end at v. Then the same argument as for case I applies, except that all edges at v are paired as arriving and leaving edges; each such pair increases $d(v)$ by 2 and therefore $d(v)$ is even.

To prove sufficiency (the 'if' or $\Leftarrow$ part) we use general induction on m, where m is the number of edges of G:

Base case: $m = 0$. Since G is connected there can be only one vertex, which is a valid (but trivial) Euler tour of length zero. So the base case is proved.

Inductive step: Suppose the result is true for all graphs with $m < M$ and let G have M edges. Since G is connected we can find a spanning tree, say, T, of G. Then T has at least one leaf (vertex of degree 1) and this has at least one adjacent edge which is not in T since no vertex of G can have

[1] often abbreviated informally as 'w.l.o.g': it means choosing a particular case does not affect the general validity of the argument.

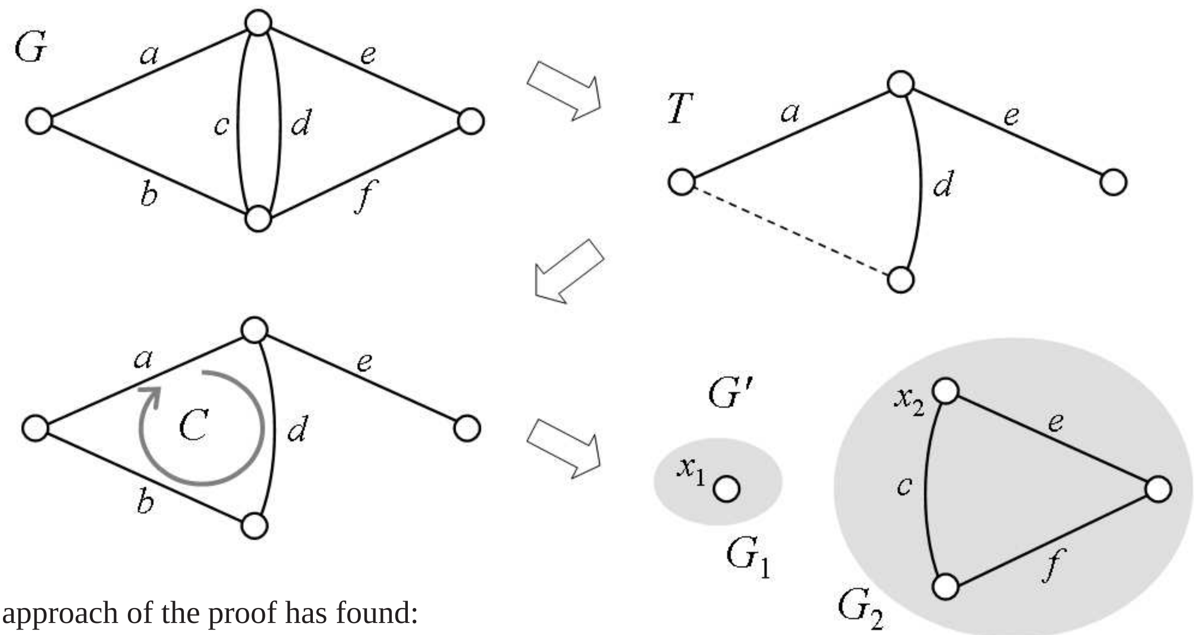
degree 1. Adding this edge to T creates a fundamental cycle, say, C. Now let G' be the graph formed by deleting all edges of C; since each vertex of C is incident with exactly two edges of C we have:

$$d_{G'}(v) = \begin{cases} d_G(v) - 2 & \text{if } v \in C \\ d_G(v) & \text{if } v \notin C \end{cases}$$

Thus every vertex has even degree in G'. However, G' may not be connected; suppose G' consists of k components $G_1, G_2, \ldots, G_k$, $k \geq 1$. By the inductive hypothesis, each component has an Euler tour: let G_i have tour S_i. Moreover, each component has at least one vertex in common with the cycle C: let $x_i \in V(C) \cap V(G_i)$. We may assume that tour S_i begins and ends at x_i, since it passes all edges in some cyclic order. Now a traversal of C, interrupted at each x_i to traverse the tour S_i will give a tour of the original graph G.

The proof now follows by induction. □

We give an example of how the proof of the Euler-Hierholzer constructs an Euler tour: it will lead us on to consider how the inductive proof may be turned into a **recursive** algorithm. We return to our original example graph:



The approach of the proof has found:

- a spanning tree T (edges a, d, e);
- an edge leaving a leaf (edge b);
- the fundamental cycle formed by b (that is $C = adb$); and
- two components formed by deleting the edges of C (component G_1 is a single vertex and has no edges; G_2 is the cycle cef).

Now G_1 has an Euler tour — the trivial tour which we may denote by $S_1 = ()$; and G_2 has an Euler tour since it is a cycle; we may write $S_2 = (cef)$. Points of intersection between G_1 and G_2 and C are x_1 and x_2 respectively, as shown. We traverse C: x_1 is encountered immediately so insert the tour $S_1 = ()$, continue around C with edge a. Now x_2 is encountered. We insert $S_2 = (cef)$ but starting at x_2 so that we actually insert the tour (efc). Finally complete C by adding the edges d and b. The complete tour of G is

$$\begin{aligned} S &= ()\, a, (e\, f\, c)\, d\, b \\ &= a\, e\, f\, c\, d\, b \quad \text{(removing brackets)} \end{aligned}$$