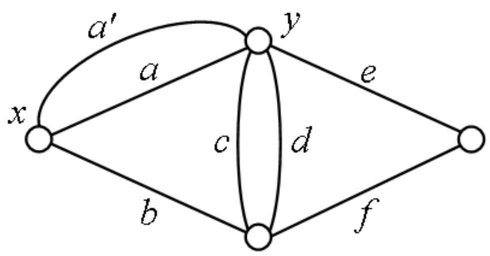
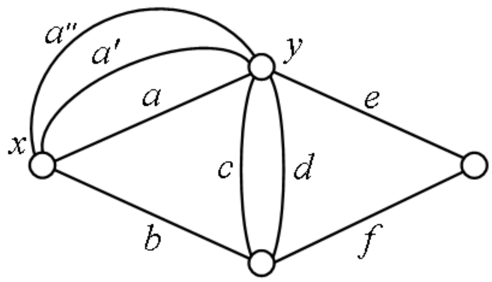
Chapter 11. Section 1

Non-Eulerian Graphs

We return to our previous example in which we added an edge a' to an Eulerian graph. This is shown below left:





Since the degrees of vertices x and y are odd, the Euler-Hierholzer Theorem tells us there can be no Euler tour.

We can restore the even degree condition by adding a second duplicate edge a'' between x and y as shown above right. Now there is an Euler tour: $a\, a'\, b\, c\, d\, f\, e\, a''$.

Observe: if we omit the last edge a'' we get a trail that uses every edge but is not closed: $a\, a'\, b\, c\, d\, f\, e$.

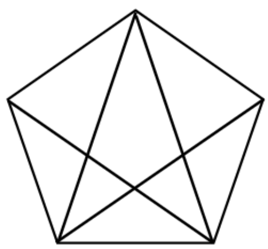
Definition In a graph G, an **Euler trail** is a trail that passes every edge of G. We say that G is **Eulerian traceable** if it admits an Euler trail.

Corollary (to Euler-Hierholzer) A graph G is Eulerian traceable if and only if there are at most two vertices of odd degree.

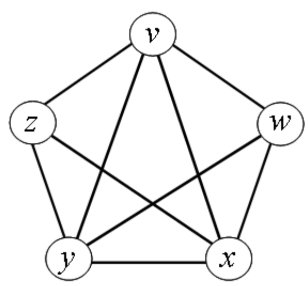
Proof: by the Handshaking Lemma, there cannot be one vertex of odd degree, so there must be zero or two. If it is zero then G is Eulerian and has a (closed) Euler trail. Suppose there are two vertices of odd degree, say, x and y. Add a new edge e from x to y. In the resulting graph, the degree of x and y is now even; the other degrees remain the same; so the graph is now Eulerian and has a tour, say, S, beginning and ending at x. We can assume that e is the final edge of S since if it is not we can cyclically permute the edges of S until it ends with e. Then $S \setminus \{e\}$ is an Euler trail of G. □

The proof of the Corollary guarantees that, in an Eulerian traceable graph with two vertices of odd degree there is an Euler trail beginning and ending in these two vertices. The proof of the necessary condition of Euler-Hierholzer means these are the **only** possible beginning and ending vertices.

Example Can you draw figure (a) below without taking your pen off the paper?



(a)

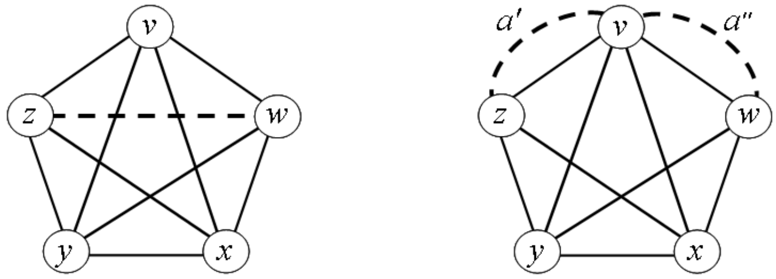


(b)

If we think of figure (a) as a graph, figure (b), then the answer is Yes: z and w are the only odd-degree vertices so the Corollary guarantees an Euler trail (which must begin and end at z and w) and this is a consecutive (no lifting the pen!) sequence of edges which includes every edge.

This solution does not help us if we need to walk round the graph in figure (b), starting and ending at, say, v, and passing every edge. A typical scenario might be post person who must deliver letters to every street in a neighbourhood before returning to the sorting office. In this case an Euler tour is what is needed — an Euler trail will not do.

The trick we used to make our earlier graph Eulerian, by adding an extra edge, is employed in figure (c) below. The graph created is the complete graph on 5 vertices, K_5. And K_n is complete for all odd values of n.



(c) (d)

This is not really a solution in the case of our post person, however — we cannot just add an extra street from z to w just in order to facilitate their return to the sorting office!

Figure (d) above proposes a different solution: we double up edges along a path that joins the two odd-degree vertices. These two vertices then acquire even degree; meanwhile the intermediate vertices on the path have their degree increased by 2 and therefore maintain their parity.

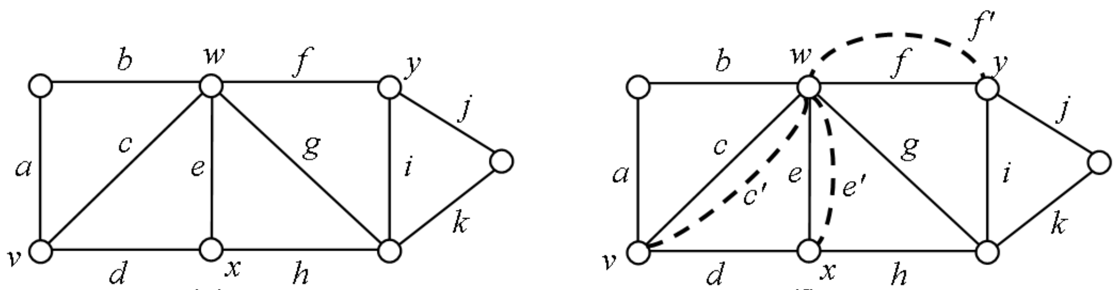
Figure (d) is Eulerian, and we can interpret an Euler tour in this graph as a closed **walk** in the original graph which repeats edges vz and vw (by using a' and a''):

Euler tour in figure (d):	vw	wx	xy	yz	zv	$\boldsymbol{a'}$	zx	xv	vy	yw	$\boldsymbol{a''}$
Closed walk passing every edge in figure (b):	vw	wx	xy	yz	zv	$\boldsymbol{vz}$	zx	xv	vy	yw	$\boldsymbol{wv}$

Since z and w are not adjacent in our graph we cannot simply double an existing edge as in our first example: the closed walk repeated two edges is the best we can do.

The general case: we have an arbitrary connected graph G. The Handshaking Lemma tells us that the odd-degree vertices are even in number: we pair them up and, for each pair, double the edges along a path in G joining the pair. The result is an Eulerian graph an Euler tour of which will correspond to a closed trail of G passing every edge.

Example: in the graph in figure (a) below, there are four vertices of odd degree: v, w, x and y:



(e) (f)

The dotted edges in figure (b) show paths joining v and w and x and y. In this graph all vertices have even degree. Notice that it does not matter that **two** paths involve w: one ends at w giving it even degree; the other passes through, preserving the parity of its degree. We have:

Euler tour in figure (b):	a	b	c	d	e	f	i	k	j	$\boldsymbol{f'}$	g	h	$\boldsymbol{e'}$	$\boldsymbol{c'}$
Closed walk passing every edge in figure (a):	a	b	c	d	e	f	i	k	j	$\boldsymbol{f}$	g	h	$\boldsymbol{e}$	$\boldsymbol{c}$

The closed walk repeats three edges: is this optimal? No, because we could have made shorter paths: doubling edges d and f would have given us an Eulerian graph and saved one repeated edge. Minimising repeats is our next goal.