Chapter 5. Section 2

Network Flows

The network $N = (V(N), A(N), s, t)$ with edge capacities given by the function $c : A(N) \to \mathbb{R}^{\geq 0}$ is a model of some application in which traffic flow from s to t is to be maximised. We need a mathematical description of 'flow'. Like capacity it will be a function: $f : A(N) \to \mathbb{R}^{\geq 0}$. This function will be subject to two constraints:

Capacity: the flow on any edge is nonnegative but cannot exceed capacity:

$$0 \leq f(e) \leq c(e), \text{ for all edges } e \in A(N); \tag{1}$$

Net flow: except for s and t, the amount of flow into a vertex is exactly matched by the flow leaving that vertex:

$$\text{for all } v \in V(N)\backslash\{s,t\}, \sum_{e \in A^-(v)} f(e) = \sum_{e \in A^+(v)} f(e), \tag{2}$$

where $A^-(v)$ and $A^+(v)$ denote the set of edges arriving at v and leaving v, respectively.

It is convenient to denote $\sum_{e \in A^-(v)}$ by $f^-(v)$ and $\sum_{e \in A^+(v)}$ by $f^+(v)$. The constraint (2) is more neatly stated as:

$$\text{for all } v \in V(N)\backslash\{s,t\}, \; f^-(v) = f^+(v).$$

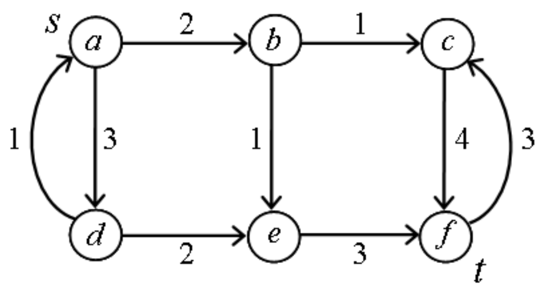
We shall see that this constraint guarantees that

$$f^+(s) - f^-(s) = f^-(t) - f^+(t). \tag{3}$$

So **net flow out of s is equal to net flow into t**. This common value is called the **value** of the flow f, and is denoted $|f|$.

Example 1: in the network on the right the flow function f is defined by:

$f(ab) = 2$	$f(ad) = 3$
$f(bc) = 1$	$f(be) = 1$
$f(cf) = 4$	$f(da) = 1$
$f(de) = 2$	$f(ef) = 3$
$f(fc) = 3$	



The flow value is

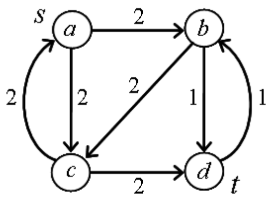
$$\begin{aligned} |f| &= f^+(s) - f^-(s) = f(ab) + f(ad) - f(da) = 2 + 3 - 1 \\ &= f^-(t) - f^+(t) = f(cf) + f(ef) - f(fc) = 4 + 3 - 3 \\ &= 4. \end{aligned}$$

Note we are concentrating on the 2nd constraint for flows here: constraint (2); we just assume, for the minute, that constraint (1) is OK.

Example 2: Continuing to ignore edge capacities, we can try to increase net flow out of s. This will have a knock-on effect which results in increased flow into t, or decreased flow out of t.

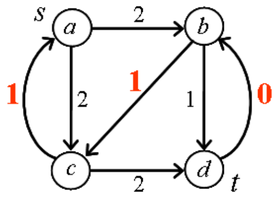
Net Flow

$f^+(s) - f^-(s)$ $\qquad$ $f^-(t) - f^+(t)$



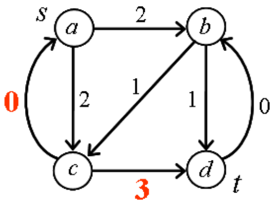
$4 - 2$ $\qquad$ $3 - 1$

$|f| = 2$



$4 - 1$ $\qquad$ $3 - 0$

$|f| = 3$



$4 - 0$ $\qquad$ $4 - 0$

$|f| = 4$

Now we shall prove that net flow out of s always equals net flow into t for a network flow function f. The main job is done by the following Lemma, which we shall also use in future lectures:

Lemma If f is an st-flow in network $N = (V(N), A(N), s, t)$ and $U \subseteq V(N)$ with $s \in U$ and $t \notin U$ then

$$f^+(U) - f^-(U) = f^+(s) - f^-(s).$$

Proof: Since f is a flow, $f^+(x) = f^-(x)$ for all $\in U\backslash\{s\}$. So

$$\begin{aligned} f^+(s) - f^-(s) &= \sum_{x\in U}\left(f^+(x) - f^-(x)\right) \quad \text{since } s \text{ is the only vertex in } U \text{ contributing to the sum} \\ &= \sum_{x\in U} f^+(x) - \sum_{x\in U} f^-(x) \quad \text{splitting the sum into two} \\ &= \quad \sigma^+ \quad - \quad \sigma^-, \text{ say, to introduce some convenient notation.} \end{aligned}$$

We claim that $\sigma^+ = f^+(U) + Q$ and $\sigma^- = f^-(U) + Q$, for some Q, and then our summation will give

$$f^+(s) - f^-(s) = \sigma^+ - \sigma^- = f^+(U) + Q - (f^-(U) + Q) = f^+(U) - f^-(U) \tag{4}$$

which is the required equality. To do this, consider every edge $e = xy \in A(N)$ and check how $f(e)$, the flow on e, contributes to σ^+ and σ^-. There are four cases:

I. $x \notin U, y \notin U$**:** then $f(e)$ is counted in neither σ^+ nor σ^-;

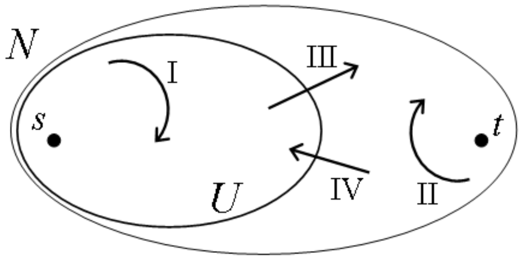
II. $x \in U, y \in U$**:** then $f(e)$ is counted in both σ^+ nor σ^- but with opposite signs; we will include this value in the quantity Q;

III. $x \in U, y \notin U$**:** then $e \in A^+(U)$, so $f(e)$ is counted in $f^+(U)$ and not in $f^-(U)$;

IV. $x \notin U, y \in U$**:** then $e \in A^-(U)$, so $f(e)$ is counted in $f^-(U)$ and not in $f^+(U)$;

and we have fulfilled the equality which we claimed in (4) and our lemma is proved. $\square$

A picture of the four cases in the proof of the lemma is the following:



The net flow for the set U is all contained in the edges of cases III and IV.

Corollary If f is an st-flow in network $N = (V(N), A(N), s, t)$ then $f^+(s) - f^-(s) = f^-(t) - f^+(t)$.

Proof: In the Lemma, put $U = V(N)\backslash\{t\}$. Then

$$\begin{aligned} f^+(s) - f^-(s) &= f^+(U) - f^-(U) \quad \text{by the Lemma} \\ &= f^-(t) - f^+(t) \quad \text{since every edge leaving/entering } U \text{ is an edge entering/leaving } t, \end{aligned}$$

and we have the required equality. □