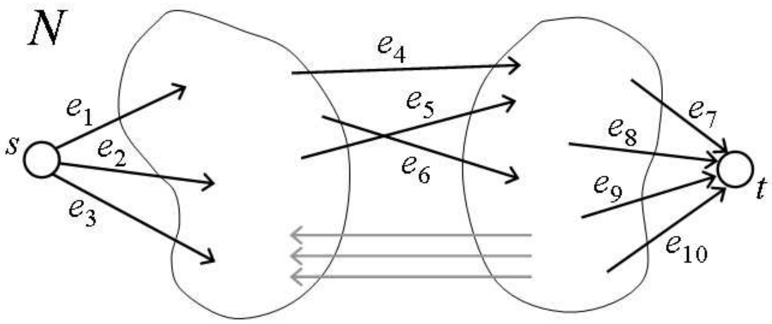
Chapter 5. Section 3

Flows vs Cuts

We now have a complete model of flow through a network N. It is time to think about *maximising* flow, that is, we want to maximise $|f| = f^+(s) - f^-(s) = f - (t) - f^+(t)$.

From now on we will assume that there are no edges entering s or leaving t, that is, $d^-(s) = d^+(t) = 0$. This is not a restriction: we know that flow into s will be balanced by flow out of t, it just simplifies things if we can assume that these flows are zero.

Then a general picture of our network N might be something like this:



Recall from Week 5, Lecture 1 that the sets $\{e_1, e_2, e_3\}$ and $\{e_7, e_8, e_9, e_{10}\}$ are *st*-cut sets since they clearly separate s from t. If we define a flow f on this network then we must have

$$|f| = f^+(s) - f^-(s) = f^+(s) = f(e_1) + f(e_2) + f(e_3) \leq c(e_1) + c(e_2) + c(e_3),$$

since $f(e) \leq c(e)$, the capacity of edge e. Similarly

$$|f| = f^-(t) - f^+(t) = f^-(t) = f(e_7) + f(e_8) + f(e_9) + f(e_{10}) \leq c(e_7) + c(e_8) + c(e_9) + c(e_{10}).$$

The middle edges e_4, e_5 and e_6 also form a cut set (the arrows going the other way across the middle are irrelevant). And again, this middle cut places a bound on the value of the flow f, namely, $|f| \leq c(e_4) + c(e_5) + c(e_6)$. We see that, in general, if $U \subseteq V(N)$, containing s and not containing t, and recalling the notation $A^+(U)$ for the set of edges leaving U,

$$|f| \leq \sum_{e \in A^+(U)} c(e).$$

Denoting the summation by $c^+(U)$, this simplifies to $|f| \leq c^+(U)$. We refer to $c + (U)$ as the **cut capacity**.

And now the whole issue of maximising network flow value is summed up in one inequality:

$$|f| \leq \min_{\substack{U \subseteq A(N) \\ s \in U,\, t \notin U}} c^+(U). \tag{1}$$

Suppose we can find a flow function f whose value $|f|$ equals a cut capacity $c^+(U)$. Then we can never do better than $|f|$ because this is what the inequality tells us; but also we need not do worse. So f maximises flow value. But then no cut capacity can go lower than $c^+(U)$, so $c^+(U)$ minimises cut capacity. This is a very important idea called **minimax**:

every flow value $\leq$ *every* cut capacity **and** value of *some* flow f = capacity of *some* cut

$\Rightarrow$ $|f|$ has been maximised and cut capacity has been minimised.

Next we shall give an algorithm which does exactly this: maximises flow by finding a flow whose value equals the capacity of a cut.