Chapter 6. Section 1

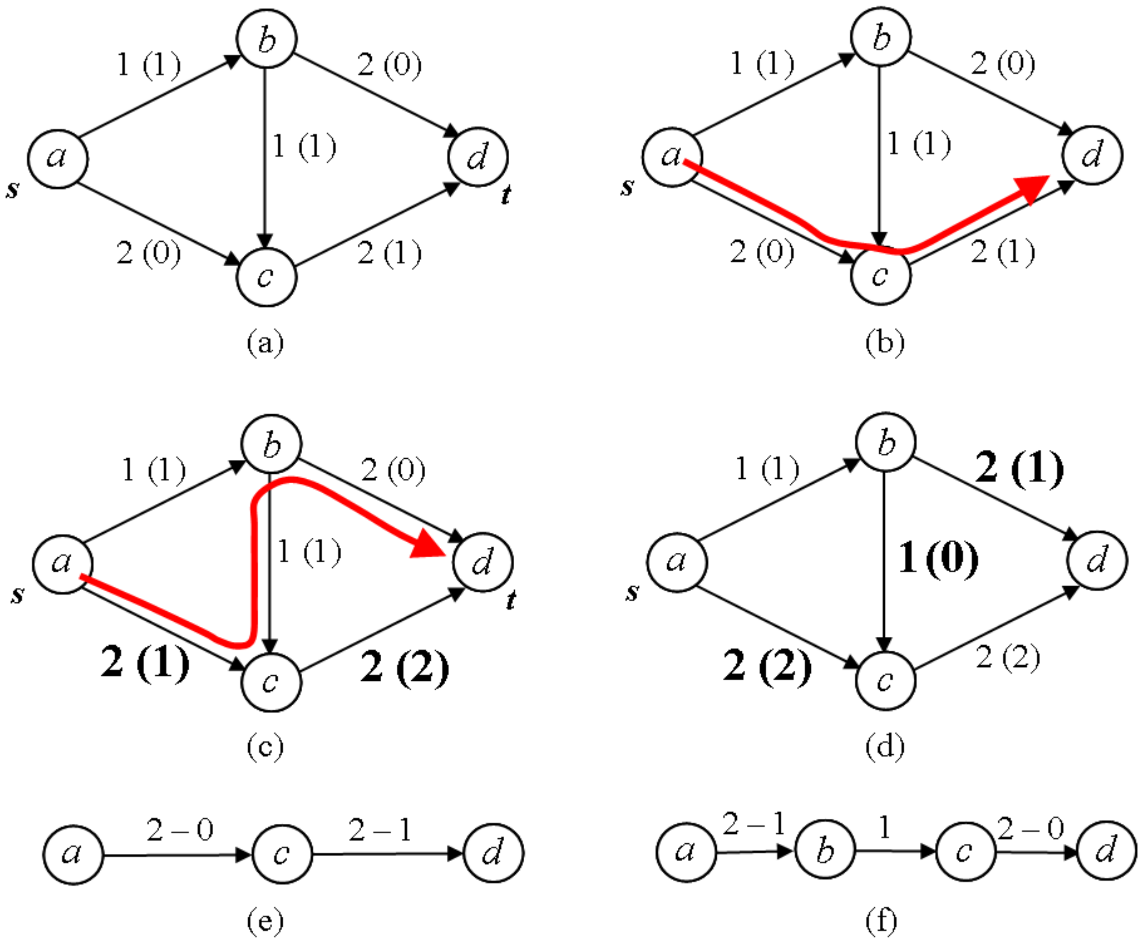
Unsaturated Paths and Trees

We will use a small example to show how we maximise flow in a network. That is, how we maximise the net flow leaving s, which is guaranteed to be the same as the net flow entering t.

In the network shown at (a) below the flow has value $|f| = 1$. But it is easy to see that we can follow a path from s to t in which all edges have spare capacity. This is shown at (b). It is called an **f-unsaturated path**. We have drawn it again at (e) with each edge e labelled with its spare capacity $c(e) - f(e)$, referred to as the **slack** on the edge.

Clearly we can increase flow along this path by the minimum slack (referred to as the slack of the path). This has been done at (c), with the altered edges in bold.

The clever bit of our algorithm is to realise we can **decrease backwards** as well as increasing forwards. The unsaturated path in (c) has spare capacity on forward edges and **nonzero**, therefore reducible, flow backwards. Again we have drawn it separately at (f) with the slack shown for each edge. Notice that slack for a backward edge e is just $f(e)$.



The second increase in flow is shown at (d): we have increased flow on forward edges and decreased it on backward edges. The flow at every vertex except s and t has increased and decreased by an equal amount, so that net flow remains zero.

Flow has now been maximised. We cannot find any further unsaturated paths from s to t; although both edges leaving vertex b have spare capacity there is no way to reach b from s with an unsaturated path.

The usual situation is that we try to find an unsaturated path from s to t by **branching out** from s with a tree, going in all possible directions, forwards (on spare capacity edges) and backwards (on nonzero flow edges). The result is called a **maximal unsaturated tree**. If it reaches t then it will contain an unsaturated path from s to t and we can increase flow; if it does **not** reach t then we can prove that our flow has been maximised for the given network.

There may be more than one possible maximal unsaturated tree. For example, in the slightly modified version of our network, shown below, there are two possible ways we can branch out from s. Neither reach t, so the flow is already maximised in this network.