Chapter 6. Section 3

The Max-Flow-Min-Cut Theorem In this lecture we will give two theorems: the first asserts the correctness of the Ford–Fulkerson algorithm; the second follows from this and says that maximum flow value = minimum cut capacity.

We need two lemmas. Recall the following Lemma from Week 5:

Lemma 1 If f is an st-flow in network $N = (V(N), A(N), s, t)$ and $U \subseteq V(N)$ with $s \in U$ and $t \notin U$ then

$$f^+(U) - f^-(U) = f^+(s) - f^-(s).$$

□

Our other lemma says that network flow $|f|$ is bounded by cut capacity.

Lemma 2 If f is an st-flow in network $N = (V(N), A(N), s, t)$ and $U \subseteq V(N)$ with $s \in U$ and $t \notin U$ then

$$|f| \le \sum_{e \in A^+(U)} c(e).$$

Proof. By definition $0 \le f(e) \le c(e)$ for all edges e.

So $f^-(U) = \sum_{e \in A^-(U)} f(e) \ge 0$ and $f^+(U) = \sum_{e \in A^+(U)} f(e) \le \sum_{e \in A^+(U)} c(e)$.

So $|f| = f^+(U) - f^-(U) \le 0 + \sum_{e \in A^+(U)} c(e) = \sum_{e \in A^+(U)} c(e)$, as required. □

Now for the correctness of Ford–Fulkerson:

Theorem If $N = (V(N), A(N), s, t)$ is a network for which the Ford–Fulkerson algorithm outputs an st-flow f and cut set C. Then $|f| = c(C)$, f is a maximum flow, and C is a minimum cut. **Proof.** $C = A^+(U)$, where U is the vertex set of a maximum f-unsaturated tree from s which does not contain t.

Now by maximality of the tree

1. $f(e) = c(e)$ for all edges leaving U, i.e. for all $e \in C = A^+(U)$, and
2. $f(e) = 0$ for all edges entering U, i.e. for all $e \in A^-(U)$.

Then we have

$$\begin{aligned} |f| &= f^+(U) - f^-(U) \quad \text{(by Lemma 1)} \\ &= \sum_{e \in A^+(U)} f(e) - \sum_{e \in A^-(U)} f(e) \quad \text{(by definition)} \\ &= \sum_{e \in A^+(U)} c(e) - 0 \quad \text{(by (1) and (2) above)} \\ &= c(C). \end{aligned}$$

So $|f| = c(C)$. And since by Lemma 2 flow is bounded by cut capacity, f must maximise flow and cut capacity must be minimised. □

Theorem (Max-Flow-Min-Cut) Theorem of the Day In any network, maximum flow value is equal to minimum cut capacity.

Proof. Apply the Ford–Fulkerson algorithm to find a maximum flow. Now the result follows from the previous theorem. □