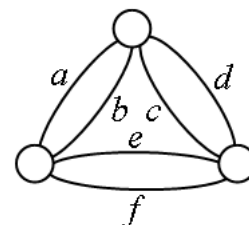


Question set for Chapter 10

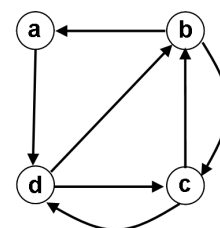
1. Prove that an Eulerian graph has a **flow**: that is we can give the edges directions and weights so that every vertex has equal weight flowing into and out of this vertex (zero net flow).
2. In an Eulerian graph G two Euler tours T_1 and T_2 are called **compatible** if no pair of edges consecutive in T_1 is also consecutive in T_2

- (a) Find two compatible Euler tours in the graph on the right.
- (b) Find three pairwise compatible Euler tours in this graph.
- (c) Find a graph on four vertices having all vertices of degree 4 (i.e. 4-regular) and **not** having three pairwise compatible Euler tours. [Hint: make one of the vertices a 'bottleneck' through which there are only 2 ways an Euler tour can pass.]



(The problem of finding a necessary and sufficient condition for a 4-regular graph to have three pairwise compatible Euler tours was solved by Bill Jackson in 1991).

3. Suppose $G = (V, E)$ is a digraph. An extension to the Euler-Hierholzer Theorem says that G has a directed closed trail which visits each edge exactly once (a *directed Euler tour*) if and only if every vertex has equal indegree and outdegree: $d^+(v) = d^-(v) = d(v)$, say, for all $v \in V$. We then say that G is an *Eulerian digraph*.



For example, in the digraph on the right, the trail $a d c d b c b a$ is a directed Euler tour.

A theorem known as the BEST Theorem (see [Theorem of the Day](#)) says that the number of distinct directed Euler tours in an Eulerian digraph G is given by the product

$$t_x(G) \prod_{v \in V} (d(v) - 1)!$$

where x is any fixed vertex of G and $t_x(G)$ is the number of spanning trees in G in which every vertex has a path directed towards x .

- (a) In the digraph above, $t_a(G) = 3$. Find the three spanning trees directed towards a .
 - (b) For the formula to be well-defined it is necessary that $t_x(G)$ has the same value for any vertex x in an Eulerian digraph G . Confirm that $t_c(G)$ also has value 3, for our directed graph G .
 - (c) Calculate the number of directed Euler tours in G using the BEST Theorem and find all the tours.
4. The graph G given below right is connected and 4-regular (every degree is 4) and is therefore Eulerian.

- (a) Apply the Recursive Euler Tour algorithm to G removing first the cycle $abca$, then the cycle $defd$ and finally the cycle $ghig$; show thereby how an Euler tour of G is built up from the cycles $adha$, $cgfc$ and $beib$ (you need **not** specify these cycles as being built up from trivial Euler tours).
- (b) Repeat part (a) but using the initial cycle $abiefcghda$

