# Solutions to question set for Chapter 10

1. Prove that an Eulerian graph has a **flow**: that is we can give the edges directions and weights so that every vertex has equal weight flowing into and out of this vertex (zero net flow).

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**SOLUTION:** Since $G$ is Eulerian it has an Euler tour. Traverse this tour. If it passes from vertex $x$ to vertex $y$ on edge $e$ then orient $e$ from $x$ to $y$. Since the tour passes every edge of $G$ by definition this gives every edge a direction. Now give every edge weight 1. Let $v$ be any vertex of $G$. The indegree of $v$ must equal its outdegree since every edge directed into $v$ by the tour must be followed by an edge directed out of $v$. So the total flow into $v$ is equal to $d^-(v) = d^+(v)$ which is the total flow out of $v$, as required.

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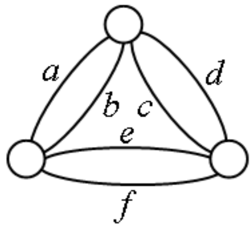
2. In an Eulerian graph $G$ two Euler tours $T_1$ and $T_2$ are called **compatible** if no pair of edges consecutive in $T_1$ is also consecutive in $T_2$

   (a) Find two compatible Euler tours in the graph on the right.

   (b) Find three pairwise compatible Euler tours in the this graph.

   (c) Find a graph on four vertices having all vertices of degree 4 (i.e. *4-regular*) and **not** having three pairwise compatible Euler tours. *[Hint: make one of the vertices a 'bottleneck' through which there are only 2 ways an Euler tour can pass.]*




(The problem of finding a necessary and sufficient condition for a 4-regular graph to have three pairwise compatible Euler tours was solved by Bill Jackson in 1991).

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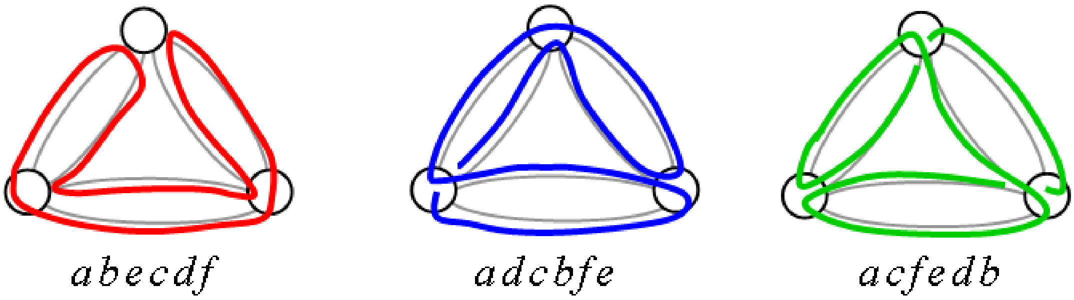
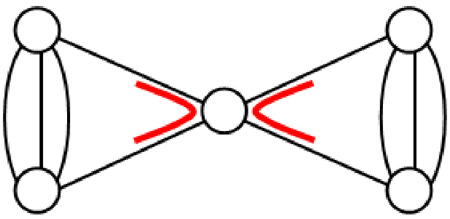
**SOLUTION:** (a) and (b) are both answered in the following diagram:




Notice that, once two compatible tours have been chosen, the third one is forced, because there are 3 ways in which consecutive pairs of edges can be chosen at any vertex.

(c) One solution is the following graph:



At each vertex there are three possible pairs of consecutive edges and all must be used in specifying three pairwise compatible Euler tours. But that means the pair shown in the above graph at the central vertex must be used. But this is impossible since no tour using these pairs of consecutive edges can pass from the left-hand side of the graph to the right-hand side.

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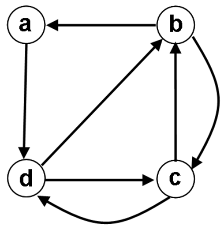
3. Suppose $G = (V, E)$ is a digraph. An extension to the Euler-Hierholzer Theorem says that $G$ has a directed closed trail which visits each edge exactly once (a *directed Euler tour*) if and only if every vertex has equal indegree and outdegree: $d^+(v) = d^+(v) = d(v)$, say, for all $v \in V$. We then say that $G$ is an *Eulerian digraph*.




For example, in the digraph on the right, the trail $a\ d\ c\ d\ b\ c\ b\ a$ is a directed Euler tour.

A theorem known as the BEST Theorem (see Theorem of the Day) says that the number of distinct directed Euler tours in an Eulerian digraph $G$ is given by the product

$$t_x(G) \prod_{v \in V} (d(v) - 1)!$$

where $x$ is any fixed vertex of $G$ and $t_x(G)$ is the number of spanning trees in $G$ in which every vertex has a path directed towards $x$.

(a) In the digraph above, $t_a(G) = 3$. Find the three spanning trees directed towards $a$.

(b) For the formula to be well-defined it is necessary that $t_x(G)$ has the same value for any vertex $x$ in an Eulerian digraph $G$. Confirm that $t_c(G)$ also has value 3, for our directed graph $G$.

(c) Calculate the number of directed Euler tours in $G$ using the BEST Theorem and find all the tours.

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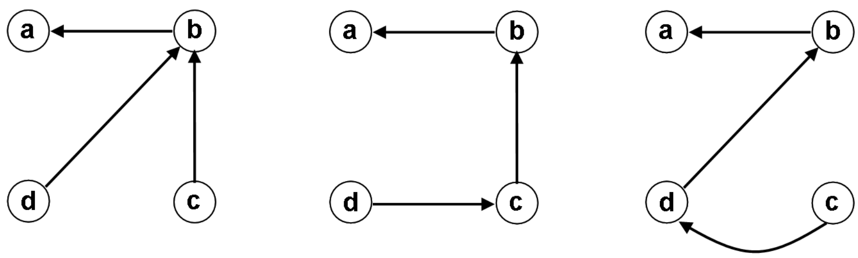
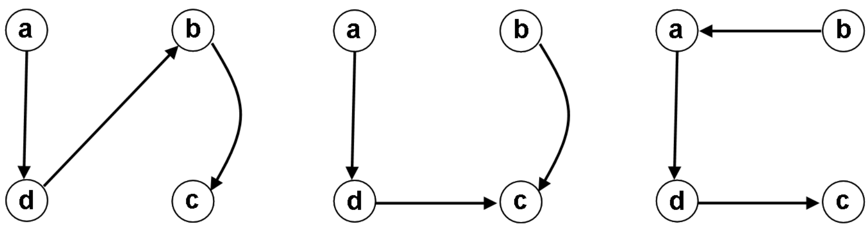
**SOLUTION:**

(a) The three spanning trees directed towards vertex $a$ are:




(b) The three spanning trees directed towards vertex $c$, confirming $t_c(G) = t_a(G)$ are:




(c) The product in the BEST Theorem is

$$3 \times (d(a) - 1)! \times (d(b) - 1)! \times (d(c) - 1)! \times (d(d) - 1)! = 3 \times 0! \times 1! \times 1! \times 1! = 3.$$

We have one tour given to us: $a\ d\ c\ d\ b\ c\ b\ a$; the other two are

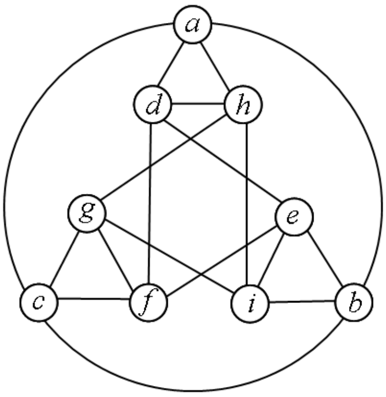
$a\ d\ b\ c\ d\ c\ b\ a$

and $a\ d\ c\ b\ c\ d\ b\ a$

4. The graph $G$ given below right is connected and 4-regular (every degree is 4) and is therefore Eulerian. (A jpeg version of this image is given on the Week 10 page of the module website.)

   (a) Apply the Recursive Euler Tour algorithm to $G$ removing first the cycle $abca$, then the cycle $defd$ and finally the cycle $ghig$; show thereby how an Euler tour of $G$ is built up from the cycles $adha$, $cgfc$ and $beib$ (you need **not** specify these cycles as being built up from trivial Euler tours).

   (b) Repeat part (a) but using the initial cycle $abiefcghda$

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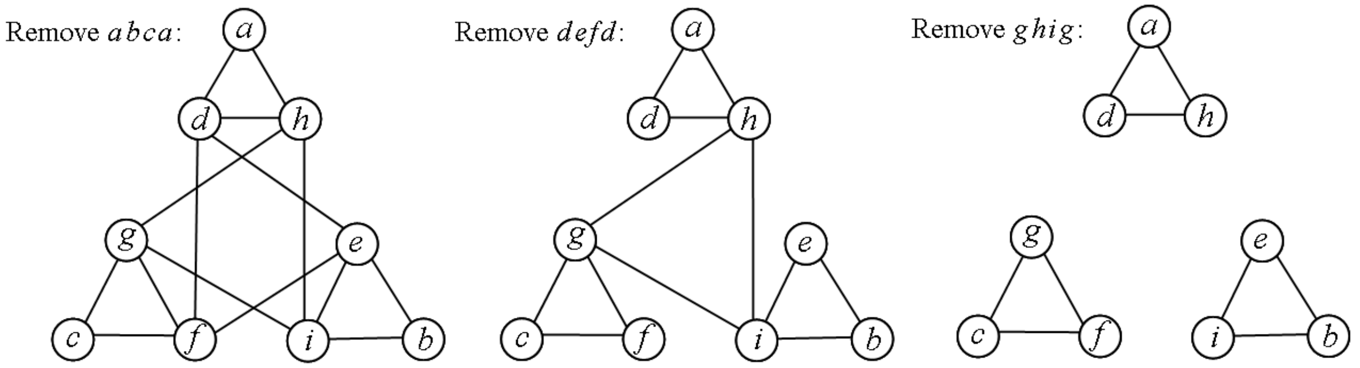
**SOLUTION:**

(a) Deleting the three cycles gives the successive drawings shown below:




Now the final drawing contains three graphs with respective Euler tours: (1) $a\,d\,h\,a$, (2) $b\,e\,i\,b$, and (3) $c\ f\ g\,c$. These must be inserted into the cycle whose removal produced the three graphs: $g\,h\,i\,g$.

Cycle round $a\,d\,h\,a$ to give $h\,a\,d\,h$.

Cycle round $b\,e\,i\,b$ to give $i\,b\,e\,i$.

Cycle round $c\ f\ g\,c$ to give $g\,c\ f\ g$.

Now these cycles may be inserted into $g\,h\,i\,g$ to give

$$g\,(c\ f\ g)\,h\,(a\,d\,h)\,i\,(b\,e\,i)\,g,$$

This must now be inserted into cycle (2): $d\,e\ f\ d$.

Cycle round $g\,(c\ f\ g)\,h\,(a\,d\,h)\,i\,(b\,e\,i)\,g$, to give $f\ g\,h\,(a\,d\,h)\,i\,(b\,e\,i)\,g\,c$.

Insert into $d\,e\ f\ d$:

$$d\,e\,(f\ g\,h\,(a\,d\,h)\,i\,(b\,e\,i)\,g\,c)\ f\,d.$$

Finally insert into $a\,b\,c\,a$.

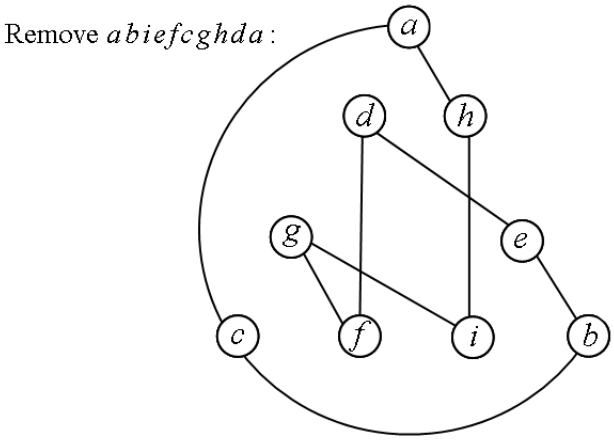
Cycle round $d\,e\,(f\ g\,h\,(a\,d\,h)\,i\,(b\,e\,i)\,g\,c)\ f\ d$ to give $(a\,d\,h)\,i\,(b\,e\,i)\,g\,c\ f\,d\,e\ f\ g\,h\,a$.

Insert into $a\,b\,c\,a$:

$$a\,b\,c\,(a\,d\,h)\,i\,(b\,e\,i)\,g\,c\ f\,d\,e\ f\ g\,h\,a$$

which is an Euler tour of the original graph.

(b) Deleting $abiefcghda$ gives:




Removing the cycle has left another complete cycle: $a\,h\,i\,g\,f\,d\,e\,b\,c\,a$. This must be inserted into the removed cycle. Since both cycles start with $a$ there is no need to cycle round. We have:

$$a\,b\,i\,e\,f\,c\,g\,h\,d\,a\,h\,i\,g\,f\,d\,e\,b\,c\,a$$

which is an Euler tour of the original graph.