Question set for Chapter 3

1. Let T be a spanning tree in a graph G and $e \in E(G)$ an edge not in T. Prove that $T \cup \{e\}$ contains a unique cycle (called a **fundamental cycle** of G). Prove that deleting any edge from a fundamental cycle yields a spanning tree of G.

2. Let G be a connected graph with weighted edges and let H be a connected subgraph of G. Consider the following Assertion:

   Assertion: Let T be a minimum-weight spanning tree for H and let $e = xy$ be an edge of minimum weight leaving H, i.e. with $x \in V(H)$ and $y \notin V(H)$. Let H^+ be the subgraph of G obtained by adding to H the vertex y and all edges of G joining H to y. Then $T + e$ is a minimum weight spanning tree for H^+.

   (a) Would this assertion, if true, give a proof of correctness of the MWST algorithm?

   (b) Is the assertion true? If so, prove it; if not produce a counterexample (a graph G and subgraph H for which we can show the assertion is false).

3. You are given a connected graph G. Consider the following three tasks:

   (a) T1: find a vertex in G of maximum degree;

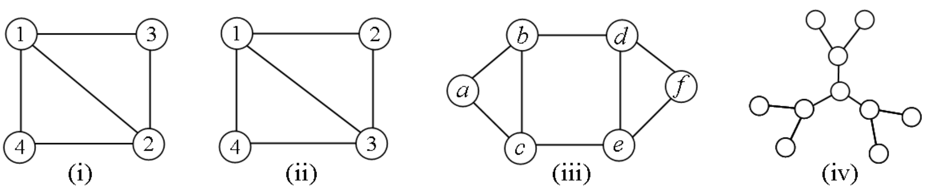
   (b) T2: decide if G is simple (i.e. has no self-loops or multiple edges);

   (c) T3: decide if G has a bridge (an edge whose deletion causes a connected graph to become disconnected).

   Give a possible value for the complexity of an algorithm for each of these tasks, using big-Oh notation and giving a brief justification for your answers.

   Are your answers different if you are told that every vertex in G has degree 2 (in which case G is said to be **2-regular**))?

4. A graph $G = (V, E)$ is said to be **prime** if we can label the vertices with the integers $1, \ldots, |V|$ in such a way that every edge has coprime endpoints (greatest common divisor 1). For example, the graph (i) below has been given a non-prime labelling since an edge joins 2 and 4 which are not coprime ing.

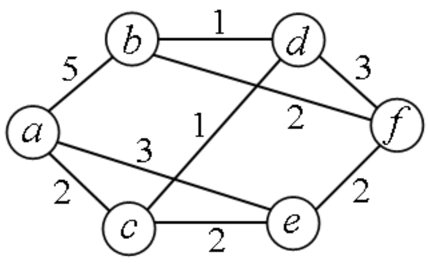
   

   (i) (ii) (iii) (iv)

   (a) Prove that the graph at (iii) is not prime.

   (b) Show that the tree at (iv) is prime.

   (A 30-year old conjecture of Entringer and Tout asserts that every tree is prime. In 2010, Penny Haxell, Oleg Pikhurko and Anusch Taraz presented a proof that this is true for all sufficiently large trees. By ‘sufficiently large’ they suggest a figure of $10^{10^{100}}$ vertices!)



5. Run Prim’s Algorithm on the graph shown on the right, starting at vertex a and showing the iterations of the algorithm (i) diagrammatically; (ii) in a table.
   Your output trees need not be identical in each case but should be clearly specified, together with their total edge weights.